A Study on Riemann Surfaces and Algebraic Curves

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Abstract

This thesis is a study on Riemann surfaces and algebraic curves.

In the first chapter, basic results, including Huwitz's formula and Plücker's formula for smooth projective plane curves are proved, with a focus on topological methods. Divisors are then introduced to organize forms, functions and maps, and the sheaf-theoretical method is emphasized throughout.

In the second chapter, a finiteness theorem is proved, leading to the existence of nonconstant meromorphic functions on any compact Riemann surface. It can be shown that every compact Riemann surface can be represented as the Riemann surface of an irreducible algebraic equation. The field of meromorphic functions on a compact Riemann surface is proved to be a finitely generated extension field of $\mathbb{C}$ of transcendence degree one. The Riemann-Roch theorem and Serre duality are then established. This chapter provides the link from the analytic category to the algebraic category and projective geometry.

In the third chapter, algebraic sheaves, invertible sheaves and line bundles are discussed. Isomorphisms between the Picard group, the group of invertible sheaves, the group of line bundles and the first Cěch cohomology group $\check{H}(X, \mathcal{O}^*_{X,alg})$ are given.

In the last chapter, a uniqueness theorem for algebraic curves is proved. Generalization of analogous theorems from $\mathbb{C}$ to finite fields may find applications to algebraic geometry codes.

摘要

本論文是研究黎曼曲面與代數曲線.

第一章節證明了一些基本結果, 包括赫爾維茨公式與平滑射影平面曲線的普呂克公式, 重點放在拓撲方法上. 然後介紹了因子, 用於組織函數, 形式和映射, 而且本文章會在各處強調層方法的使用.

第二章節證明了 finiteness 定理, 證明了在緊黎曼曲面上存在不恒定亞純函數. 每個緊黎曼曲面可看成不可約代數方程的黎曼曲面. 在緊黎曼曲面上, 亞純函數的域是複數平面的有限生成擴張域, 超越次數為一. 黎曼洛赫定理與塞爾對偶定理提供了緊黎曼曲面與代數範疇和投射範疇之間的聯繫.

第三章節討論了代數層, 可逆層和線叢, 而且證明了皮卡群, 代數層的群, 可逆層的群, 線叢的群和非零全純函數代數層的 Cěch 第一上同島是同構的.

最後一個章節證明了代數曲線的唯一性定理. 此定理若能推廣到有限域上, 可能在代數幾何密碼有所應用.

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Chapter 1

Basic Notions of Riemann Surfaces

1.1 Functions, Forms and Hurwitz's Formula

Definition 1.1 *A Riemann surface is a second countable connected Hausdorff topological space X such that there is an open cover $\{U_\alpha\}$ of X, with homeomorphisms $\phi_\alpha : U_\alpha \to V_\alpha$ where V_α is an open set in $\mathbb{C}$ and ϕ_α are compatible to each other, that is $\phi_\alpha \circ \phi_\beta^{-1}$ is holomorphic on $U_\alpha \bigcap U_\beta$.*

Some basic examples of compact Riemann surfaces are the projective line, the Riemann sphere and complex tori. Other examples include smooth irreducible affine plane curves and smooth projective plane curves.

Definition 1.2 *An affine plane curve X is the locus of zeros in $\mathbb{C}^2$ of a polynomial $f(z,w)$. The polynomial $f(z,w)$ is singular at p if $\dfrac{\partial f}{\partial z}(p) = \dfrac{\partial f}{\partial w}(p) = 0$. The affine plane curve X is smooth if f is non-singular at each point of X.*

By the implicit function theorem, we can define complex charts on a smooth affine plane curve X. Furthermore, if $f(z,w)$ is an irreducible polynomial, then

X is connected. Hence every smooth irreducible affine plane curve is a Riemann surface. Note that no affine plane curve is compact since it is not bounded in $\mathbb{C}^2$.

Definition 1.3 *A projective plane curve X is the locus of zeros in $\mathbb{P}^2$ of a homogenous polynomial $F(x, y, z)$, that is $X = \{[x : y : z] \in \mathbb{P}^2 : F(x, y, z) = 0\}$. A point $p \in X$ is singular if $\frac{\partial F}{\partial x}(p) = \frac{\partial F}{\partial y}(p) = \frac{\partial F}{\partial z}(p) = 0$. The curve X is called smooth if it is non-singular at each point of X. The degree of a projective plane curve defined by F is the degree of the homogenous polynomial F.*

Note that every non-singular homogenous polynomial is irreducible. Suppose X is a smooth projective plane curve defined by a non-singular homogenous polynomial F. When X is restricted to $x \neq 0$, $y \neq 0$, $z \neq 0$ respectively, it is a smooth irreducible affine plane curve. Also, X is compact since it is a closed subset in $\mathbb{P}^2$. Therefore, every smooth projective plane curve is a compact Riemann surface.

Definition 1.4 *Let X be a Riemann surface where $X \subset \mathbb{P}^n$. X is a smooth projective curve if at each $p \in X$, there exists a homogenous coordinate z_j such that*

1. *$z_j \neq 0$ at p*
2. *for all k, $\frac{z_k}{z_j}$ is a holomorphic function on X near p*
3. *there exists a homogenous coordinate z_i such that $\frac{z_i}{z_j}$ is a local coordinate on X near p*

Any ratio of homogenous polynomials of the same degree where the denominator not identically zero is a meromorphic function on a smooth projective curve.

Definition 1.5 *Let X be a Riemann surface. Let f be a complex-valued function defined on a neighborhood W of $p \in X$. f is holomorphic at p if there exists a* *chart $\phi : U \to V$ with $p \in U$, such that $f \circ \phi^{-1}$ is holomorphic at $\phi(p)$. f is a holomorphic function on X if f is holomorphic at each point of X. Denote $\mathcal{O}(W) = \{f : W \to \mathbb{C}$ where f is holomorphic on $W\}$.*

By the maximum modulus theorem, if f is holomorphic on a compact Riemann surface X, then f is a constant function. To study compact Riemann surfaces, we need to study meromorphic functions.

Definition 1.6 *Let X be a Riemann surface. Let f be a complex-valued function defined on a neighborhood W of $p \in X$. f is meromorphic at p if there exists a chart $\phi : U \to V$ with $p \in U$, such that $f \circ \phi^{-1}$ is meromorphic at $\phi(p)$. f is a meromorphic function on X if f is meromorphic at every point of X. Denote $\mathcal{M}(W) = \{f : W \to \mathbb{C}$ where f is meromorphic on $W\}$.*

The order of a meromorphic function f at p, denoted by $\mathrm{ord}_p(f)$, is the order of its Laurent series in a local coordinate. This is independent of the choice of local coordinate. For every nonconstant meromorphic function f on a compact Riemann surface X, $\sum_{p \in X} \mathrm{ord}_p(f) = 0$.

Definition 1.7 *A mapping between two Riemann surfaces $F : X \to Y$ is holomorphic at $p \in X$ if there exist charts $\phi_1 : U_1 \to V_1$ on X with $p \in U_1$ and $\phi_2 : U_2 \to V_2$ on Y with $F(p) \in U_2$, such that $\phi_2 \circ F \circ \phi_1^{-1}$ is holomorphic at $\phi_1(p)$. F is a holomorphic map if F is holomorphic at each point of X.*

Fix a nonconstant holomorphic map $F : X \to Y$. For every $p \in X$, there is a unique integer $m \geq 1$ such that for every chart $\phi_2 : U_2 \to V_2$ on Y centered at $F(p)$, there exists a chart $\phi_1 : U_1 \to V_1$ on X centered at p such that $\phi_2 \circ F \circ \phi_1^{-1}(z) = z^m$. The integer m is called the multiplicity of F at p, denoted by $\mathrm{mult}_p(F)$. A point $p \in X$ is a ramification point of F if $\mathrm{mult}_p(F) \geq 2$. A point $q \in Y$ is a branch point of F if it is the image of a ramification point of F. The degree of F, denoted by $\deg(F)$, is the sum of multiplicities of F at the preimages of $q \in Y$, that is $\deg(F) = \sum_{p \in F^{-1}(q)} \mathrm{mult}_p(F)$. This is independent of $q \in Y$.

Theorem 1.8 (Hurwitz's Formula) *Let $F : X \to Y$ be a nonconstant holomorphic map between two compact Riemann surfaces. Then $2g(X) - 2 = \deg(F)$ $(2g(Y) - 2) + \sum_{p\in X}(\mathrm{mult}_p(F) - 1)$, where $g(X)$ and $g(Y)$ are the genus numbers of X and Y respectively.*

Proof Since X is compact, there are finite numbers of ramification points. Take a triangulation of Y such that each branch point is a vertex, and lift the triangulation via the map F. Then every ramification point is a vertex of the triangulation on X. Let v, e, t be the numbers of vertices, edges, triangles respectively in the triangulation of Y, and $\tilde{v}, \tilde{e}, \tilde{t}$ be those on X. Since there is no ramification point inside any triangle of X, $\tilde{e} = \deg(F)e$ and $\tilde{t} = \deg(F)t$. Also, $\tilde{v} = \sum_{q \textit{ vertex of } Y} \sum_{p\in F^{-1}(q)} 1 = \sum_{q \textit{ vertex of } Y} (\deg(F) + \sum_{p\in F^{-1}(q)}(1 - \mathrm{mult}_p(F)) = \deg(F)v + \sum_{p \textit{ vertex of } X}(1 - \mathrm{mult}_p(F))$.

By the theory of the Euler number, $2 - 2g(X) = \tilde{v} - \tilde{e} + \tilde{t} = \deg(F)v + \sum_{p \textit{ vertex of} X}(1 - \mathrm{mult}_p(F)) - \deg(F)e + \deg(F)t = \deg(F)(v - e + t) + \sum_{p \textit{ vertex of } X}$ $(1 - \mathrm{mult}_p(F))$. Hence $2g(X) - 2 = \deg(F)(2g(Y) - 2) + \sum_{p\in X}(\mathrm{mult}_p(F) - 1)$. □

Definition 1.9 *Let X be a Riemann surface. A holomorphic (meromorphic) 1-form on X is a collection of holomorphic (meromorphic) 1-forms, one for each chart on X, such that if $\omega_1 = f(z)dz$ for one chart and $\omega_2 = g(w)dw$ for another chart, then $g(w) = f(T(w))T'(w)$ where $z = T(w)$. Denote the set of holomorphic 1-forms by $\Omega^{(1)}(X)$, and the set of meromorphic 1-forms by $\mathcal{M}^{(1)}(X)$*

Definition 1.10 *Let X be a Riemann surface. A C^∞ 1-form on X is a collection of C^∞ 1-forms, one for each chart on X, such that if $\omega_1 = f_1(z, \overline{z})dz + g_1(z, \overline{z})d\overline{z}$ for one chart and $\omega_2 = f_2(w, \overline{w})dz + g_2(w, \overline{w})d\overline{w}$ for another chart, then $f_2(w, \overline{w}) = f_1(T(w), \overline{T(w)})T'(w)$ and $g_2(w, \overline{w}) = g_1(T(w), \overline{T(w)})T'(w)$ where $z = T(w)$. Denote the set of C^∞ 1-forms by $\mathcal{E}^{(1)}(X)$.*

Since dz and $d\overline{z}$ parts of a C^∞ 1-form are preserved under the change of coordinates, we can define the followings.

Definition 1.11 *Let X be a Riemann surface. A C^∞ 1-form on X is of type $(1,0)$ if it is locally in the form of $f(z,\overline{z})dz$. It is of type $(0,1)$ if it is locally in the form of $g(z,\overline{z})d\overline{z}$. Denote the set of C^∞ 1-forms of type $(1,0)$ by $\mathcal{E}^{(1,0)}(X)$, and the set of C^∞ 1-forms of type $(0,1)$ by $\mathcal{E}^{(0,1)}(X)$.*

Let ω be a meromorphic 1-form on a Riemann surface X. Fix $p \in X$, choose a local coordinate z centered at p and write $\omega = f(z)dz = (\sum_{i=n}^{\infty} c_i z^i)dz$ near p. The order of ω at p, denoted by $\mathrm{ord}_p(\omega)$, is the order of the meromorphic function f at p. The residue of ω at p, denoted by $\mathrm{Res}_p(\omega)$, is the coefficient c_{-1}. Observe that $\mathrm{ord}_p(\omega)$ and $\mathrm{Res}_p(\omega)$ are independent of the choice of local coordinates. The residue theorem says that if ω is a meromorphic 1-form on a compact Riemann surface X, then $\sum_{p\in X}\mathrm{Res}_p(\omega) = 0$.

Let $F : X \to Y$ be a nonconstant holomorphic map between two compact Riemann surfaces. For any chart $\phi : U \to V$ on X with local coordinate w and $\psi : U' \to V'$ on Y with local coordinate z, such that $F(U) \subseteq U'$. Write F in terms of the local coordinates, $z = h(w)$ for some holomorphic function h. Let ω be a meromorphic 1-form on Y, and write $\omega = f(z)dz$. Define $F^*\omega = f(h(w))h'(w)dw$. Then $F^*\omega$ gives a well-defined meromorphic 1-form on X. $F^*\omega$ is called the pullback of ω via F.

By choosing local coordinates w centered at p and z centered at $F(p)$, a meromorphic 1-form ω on Y is in the form of (cz^n+ higher order terms)dz where $n = \mathrm{ord}_{F(p)}(\omega)$. Let $m = \mathrm{mult}_p(F)$. Then $F^*\omega$ is in the form of (cw^{nm}+ higher order terms)$(mw^{m-1})dw$. Hence the order of $F^*\omega$ is $nm + m - 1$, that is $\mathrm{ord}_p(F^*(\omega)) = (1+ \mathrm{ord}_{F(p)}(\omega))\mathrm{mult}_p(F) - 1$.

1.2 Divisors

Definition 1.12 *Let X be a Riemann surface. A divisor on X is a function $D : X \to \mathbb{Z}$ whose support is a discrete subset of X.*

Set of divisors on X, denoted by $\mathrm{Div}(X)$, is an abelian group under pointwise addition. Write a divisor D as a formal sum, $D = \sum_{p\in X} D(p)\cdot p$. If X is a compact Riemann surface, then divisors on X have finite support. Hence, we can define the degree of a divisor D on a compact Riemann surface X to be $\deg(D) = \sum_{p\in X} D(p)$.

Definition 1.13 *Let X be a Riemann surface and f be a meromorphic function on X. The divisor of f,* $\mathrm{div}(f) = \sum_{p\in X}\mathrm{ord}_p(f)\cdot p$ *is called a principal divisor on X. Denote* $\mathrm{PDiv}(X)$ *as the set of principal divisors on X.*

Note that $\mathrm{PDiv}(X)$ is a subgroup of $\mathrm{Div}(X)$. The group of divisors modulo principal divisors is called the Picard group of X, denoted by $\mathrm{Pic}(X)$, that is $\mathrm{Pic}(X) = \mathrm{Div}(X)/\mathrm{PDiv}(X)$. Also note that the degree of any principal divisor on a compact Riemann surface is zero because $\deg(\mathrm{div}(f)) = \sum_{p\in X}\mathrm{ord}_p(f) = 0$ for any meromorphic function f on a compact Riemann surface X.

Definition 1.14 *Let $F : X \to Y$ be a nonconstant holomorphic map between two Riemann surfaces. The ramification divisor of F, denoted by R_F, is the divisor on X defined by $R_F = \sum_{p\in X}[\mathrm{mult}_p(F) - 1]\cdot p$. The branch divisor of F, denoted by B_F, is the divisor on Y defined by $B_F = \sum_{q\in Y}[\sum_{p\in F^{-1}(q)}(\mathrm{mult}_p(F) - 1)]\cdot q$.*

Observe that $\deg(R_F)$ is precisely the error term in Hurwitz's formula. That is $2g(X) - 2 = \deg(F)(2g(Y) - 2) + \deg(R_F)$.

Definition 1.15 *Let X be a Riemann surface and ω be a nonzero meromorphic 1-form on X . The divisor of ω,* $\mathrm{div}(\omega) = \sum_{p\in X}\mathrm{ord}_p(\omega)\cdot p$ *is called a canonical divisor on X. Denote* $\mathrm{KDiv}(X)$ *as the set of canonical divisors on X.*

Definition 1.16 *Two divisors on a Riemann surafce X are said to be linearly equivalent, that is $D_1 \sim D_2$, if their difference is a principal divisor.*

Note that the above relation is an equivalence relation on $\mathrm{Div}(X)$. A equivalence class is a coset for $\mathrm{PDiv}(X)$. Also, if X is compact, linearly equivalent divisors have the same degree because the degree of any principal divisor on a compact Riemann surface is zero.

Lemma 1.17 *Let ω_1 and ω_2 be two meromorphic 1-forms on a Riemann surface X and $\omega_1 \not\equiv 0$. Then there exists a meromorphic function f on X such that $\omega_2 = f\omega_1$.*

Proof For any chart $\phi : U \to V$ on X with local coordinate z, and for $i = 1, 2$, write $\omega_i(z) = g_i(z)dz$ for some meromorphic functions g_i on V. Let $f = \dfrac{g_2}{g_1} \circ \phi$. Then f is a meromorphic function on U. It suffices to show that f agrees on intersection. Suppose w is another local coordinate with $z = T(w)$. Write $\omega_i(w) = \widetilde{g_i}(w)dw$ where $\widetilde{g_i}(w) = g_i(T(w))T'(w)$. Then $\dfrac{\widetilde{g_2}(w)}{\widetilde{g_1}(w)} = \dfrac{g_2(T(w))T'(w)}{g_1(T(w))T'(w)} = \dfrac{g_2(z)}{g_1(z)}$. Hence there exists a global meromorphic function f such that $\omega_1 = f\omega_2$. □

Proposition 1.18 *Let X be a compact Riemann surface of genus g with a non-constant meromorphic function. Then every canonical divisor of a non-vanishing meromorphic 1-form on X has degree $2g - 2$.*

Proof Let f be a nonconstant meromorphic function on X. Consider the holomorphic map $F : X \to \mathbb{C}_\infty$ defined by $F(p) = f(p)$ if p is not a pole of f, and $F(p) = \infty$ if p is a pole of f. Consider the meromorphic 1-form $\omega = dz$ on $\mathbb{C}_\infty$. It has double poles at ∞ with no other pole and zero, thus it has degree -2. Then $\mathrm{div}(F^*(\omega))$ is a canonical divisor on X with $\mathrm{ord}_p(F^*(\omega)) = (1+ \mathrm{ord}_{F(p)}(\omega))\mathrm{mult}_p(F)-1$. Thus $\deg(\mathrm{div}(F^*(\omega)) = \sum_{p\in X}\mathrm{ord}_p(F^*(\omega)) = \sum_{p\in X}[(1+ \mathrm{ord}_{F(p)}(\omega))\mathrm{mult}_p(F)-1] = \sum_{p\in F^{-1}(q),\ q\neq\infty}(\mathrm{mult}_p(F)-1)+\sum_{p\in F^{-1}(\infty)}(-\mathrm{mult}_p(F) -1) = \sum_{p\in X}(\mathrm{mult}_p(F) - 1) - \sum_{p\in F^{-1}(\infty)}(2\mathrm{mult}_p(F))$.

By Hurwitz's formula, $\sum_{p\in X}[\mathrm{mult}_p(F) - 1] = 2g - 2 + 2\deg(F)$. Hence, $\deg(\mathrm{div}(F^*(\omega)) = 2g - 2 + 2\deg(F) - 2\deg(F) = 2g - 2$. By Lemma 1.17, every canonical divisor of a non-vanishing meromorphic 1-form on X has degree $2g-2$ since degree of every principal divisor is zero on a compact Riemann surface X. $\square$

Definition 1.19 *Let D be a divisor on a Riemann surface X. $L(D) = \{f \in \mathcal{M}(X) : \operatorname{div}(f) \geq -D\}$ is the space of meromorphic functions with poles bounded by D. $L^{(1)}(D) = \{\omega \in \mathcal{M}^{(1)}(X) : \operatorname{div}(\omega) \geq -D\}$ is the space of meromorphic 1-forms with poles bounded by D.*

Note that $L(D)$ and $L^{(1)}(D)$ are complex vector spaces.

Suppose $D_1 \sim D_2$. Write $D_1 = D_2 +$ $\operatorname{div}(f)$ for some meromorphic function f. Consider the multiplication map from $L(D_1)$ to $L(D_2)$, by sending $g \in L(D_1)$ to $fg \in L(D_2)$. Inverse of this map is the multiplication map from $L(D_2)$ to $L(D_1)$, by sending $g \in L(D_2)$ to $\frac{g}{f} \in L(D_1)$. This gives an isomorphism between $L(D_1)$ and $L(D_2)$. Therefore, if $D_1 \sim D_2$, then $L(D_1) \cong L(D_2)$. Also note that, similarly, if $D_1 \sim D_2$, then $L^{(1)}(D_1) \cong L^{(1)}(D_2)$.

Proposition 1.20 *Let D be a divisor and K be a canonical divisor on a Riemann surface X. Then there is an isomorphism between the two complex vector spaces $L^{(1)}(D)$ and $L(D+K)$.*

Proof Write $K = \operatorname{div}(\omega)$ for some meromorphic 1-form ω on X. Define a map $\mu_\omega : L(D+K) \to L^{(1)}(D)$ by $\mu_\omega(f) = f\omega$. For $f \in L(D+K)$, $\operatorname{div}(f) \geq -D-K$. Then $\operatorname{div}(f\omega) = \operatorname{div}(f) +$ $\operatorname{div}(\omega) = \operatorname{div}(f) + K \geq -D$, thus $f\omega \in L^{(1)}(D)$. Hence μ_ω is well-defined. The map is clearly $\mathbb{C}$-linear and one-to-one. It remains to show that it is onto. Let $\eta \in L^{(1)}(D)$. By Lemma 1.17, there exists a meromorphic function g such that $\eta = g\omega$. Then $\operatorname{div}(g) + D + K = \operatorname{div}(g) + D +$ $\operatorname{div}(\omega) = \operatorname{div}(g\omega) + D = \operatorname{div}(\eta) + D \geq 0$. Therefore, there exists $g \in L(D+K)$ such that $\mu_\omega(g) = \eta$. $\square$

1.3 Plücker's Formula for a Smooth Projective Plane Curve

Let X be a smooth projective curve. Let $G(x_0, \ldots, x_n)$ be a nonzero homogeneous polynomial on X. Fix $p \in X$ such that $G(p) = 0$. Choose a homogenous polynomial H of the same degree as G, but $H(p) \neq 0$. Hence $\frac{G}{H}$ is a meromorphic function on X such that $\frac{G}{H}(p) = 0$. Define $\operatorname{div}(G)(p) = \operatorname{ord}_p(\frac{G}{H})$ where $G(p) = 0$. At point p where $G(p) \neq 0$, set $\operatorname{div}(G)(p) = 0$. Suppose we choose another homogenous polynomial H'. Then $\frac{G}{H'} = \frac{G}{H}\frac{H}{H'}$. Also $\frac{G}{H}$ and $\frac{G}{H'}$ have the same order at p because $\frac{H}{H'}$ has order zero at p. Thus $\operatorname{div}(G)$ is independent of the choice of non-vanishing polynomial H, and $\operatorname{div}(G)$ is well-defined.

Definition 1.21 *Let G be a nonzero homogeneous polynomial on a smooth projective curve X. Then $\operatorname{div}(G)$ defined above is called the intersection divisor of G on X. When G is degree one, $\operatorname{div}(G)$ is called the hyperplane divisor of G on X.*

Observe that any intersection divisor is strictly positive because G vanishes at p but H does not vanish at p.

Lemma 1.22 *Let X be a smooth projective curve. If G_1 and G_2 are two homogenous polynomials with the same degree, then the two intersection divisors $\operatorname{div}(G_1)$ and $\operatorname{div}(G_2)$ on X are linearly equivalent and have the same degree.*

Proof Let G_1, G_2 be two homogeneous polynomials of the same degree on X. Fix $p \in X$, choose a homogenous polynomial H of the same degree as G_1, G_2 such that $H(p) \neq 0$. Then $\operatorname{div}(G_i)(p) = \operatorname{ord}_p(\frac{G_i}{H})$ for $i = 1, 2$. Let $f = \frac{G_1}{G_2} = \frac{G_1}{H}/\frac{G_2}{H}$ which is a meromorphic function on X. $\operatorname{ord}_p(f) = \operatorname{ord}_p(\frac{G_1}{H}) - \operatorname{ord}_p(\frac{G_2}{H})$. Then $\operatorname{div}(f) = \operatorname{div}(G_1) - \operatorname{div}(G_2)$. Hence the two intersection divisors $\operatorname{div}(G_1)$ and $\operatorname{div}(G_2)$ on X are linearly equivalent. Since X is compact, linearly equivalent divisors have the same degree. □

By Lemma 1.22, we have the following definition.

Definition 1.23 *Let X be a smooth projective curve. Then* $\deg(X)$ *is the degree of any hyperplane divisor on X.*

If X is a smooth projective plane curve defined by the homogenous polynomial $F(x, y, z) = 0$, then the above definition of the degree of X coincides with the degree of the homogenous polynomial F.

Proposition 1.24 *Let X be a smooth projective plane curve defined by the homogenous polynomial $F(x, y, z) = 0$ where F has degree d. Then X has degree d in the sense that any hyperplane divisor on X has degree d.*

Proof Let G be a homogenous polynomial of degree one. Then $\operatorname{div}(G)$ is a hyperplane divisor. We assume that $G(x, y, z) = x$ and $[0 : 0 : 1] \notin X$ by changing coordinates. Consider the homogenous polynomial y. Then x and y do not have a common zero on X because $[0 : 0 : 1] \notin X$. Also, $\operatorname{div}(G) = \sum_{ord_p(\frac{x}{y})>0} \operatorname{ord}_p(\frac{x}{y}) \cdot p$. Let $H : X \to \mathbb{C}_\infty$ be the associated holomorphic map to the meromorphic function $\frac{x}{y}$. Then $\operatorname{div}(G) = \sum_{ord_p(\frac{x}{y})>0} \operatorname{ord}_p(\frac{x}{y}) \cdot p = \sum_{p \in H^{-1}(0)} \operatorname{mult}_p(H) \cdot p$, because $\operatorname{mult}_p(H) = \operatorname{ord}_p(\frac{x}{y})$ for $\frac{x}{y}(p) = 0$.

To compute $\deg(\operatorname{div}(G))$, it suffices to compute $\deg(H)$. Fix $c \in \mathbb{C}$. For $H(p) = c$, write $p = [x : y : z] \in X$, then $x = cy$ and $F(p) = 0$. If $c \neq 0$, then $x \neq 0$ and $y \neq 0$ since $[0 : 0 : 1] \notin X$. Thus points in $H^{-1}(c)$ is in the form $[c : 1 : z]$ with $F(c, 1, z) = 0$. For a general c, that means c is not a branch point of H, $F(c, 1, z) = 0$ is a polynomial in z of degree d and has d distinct solutions. So, $\deg(H) = d$. $\square$

Theorem 1.25 (Bezout's Theorem) *Let X be a smooth projective curve of degree d. Let G be a homogenous polynomial of degree e, not identically zero on X. Then* $\deg(\operatorname{div}(G)) = e\deg(X) = ed$.

Proof Let H be a homogenous polynomial of degree one, defining a hyperplane divisor $\mathrm{div}(H)$ on X. H^e has degree e. Thus we have two intersection divisors $\mathrm{div}(H^e)$ and $\mathrm{div}(G)$. By Lemma 1.22, $\deg(\mathrm{div}(H^e)) = \deg(\mathrm{div}(G))$. Also, $\mathrm{div}(H^e) = e\mathrm{div}(H)$ implies $\deg(\mathrm{div}(H^e)) = e\deg(\mathrm{div}(H))$. Therefore, $\deg(\mathrm{div}(G)) = e\deg(\mathrm{div}(H)) = e\deg(X)$. □

Lemma 1.26 *Let X be a smooth projective plane curve defined by the homogenous polynomial $F(x, y, z) = 0$. Consider $\pi : X \to \mathbb{P}^1$ by $\pi[x : y : z] = [x : z]$. Note that $\frac{\partial F}{\partial y}$ is also a homogenous polynomial. Then intersection divisor $\mathrm{div}(\frac{\partial F}{\partial y})$ on X is exactly the ramification divisor $R_\pi = \sum_{p\in X}(\mathrm{mult}_p(\pi) - 1) \cdot p$ of π, that is $R_\pi = \mathrm{div}(\frac{\partial F}{\partial y})$.*

Proof It suffices to prove the statement on the open set $X \cap U$ where $U = \{[x : y : z] \in \mathbb{P}^2 : z \neq 0\}$. Then $X \cap U$ is isomorphic to the affine plane curve defined by $f(x, y) = F(x, y, 1) = 0$. Also $\pi(x, y) = x$ on $X \cap U$. Suppose $\frac{\partial f}{\partial y}(p) \neq 0$ where $p \in X \cap U$. Then π is a chart at p and so it has multiplicity one. Suppose $\frac{\partial f}{\partial y}(p) = 0$. Then $\frac{\partial f}{\partial x}(p) \neq 0$ since X is smooth at p. So, y is a local coordinate for X near p. By the implicit function theorem, near p, X is locally a graph of holomorphic function $g(y)$. Thus $f(g(y), y) \equiv 0$ in a neighborhood of y_0 where $p = (x_0, y_0)$, and $\frac{\partial f}{\partial x}g'(y) + \frac{\partial f}{\partial y} \equiv 0$ where $g(y)$ is exactly the local formula for π. Since $\frac{\partial f}{\partial x}(p) \neq 0$ and $\frac{\partial f}{\partial y}(p) = 0$, $g'(y_0) = 0$. Since $\mathrm{ord}_p(g'(y)) = \mathrm{mult}_p(\pi) - 1$, π is ramified at $p \in X$. Therefore, π is ramified at $p \in X \cap U$ if and only if $\frac{\partial f}{\partial y}(p) = 0$.

Now suppose p is a point of ramification for π and is a zero of $\frac{\partial f}{\partial y}$. Then by above, $\frac{\partial f}{\partial x}g'(y) + \frac{\partial f}{\partial y} \equiv 0$ and $\frac{\partial f}{\partial x}(p) \neq 0$. Hence $\mathrm{ord}_p(\frac{\partial f}{\partial y}) = \mathrm{ord}_p(g'(y)) = \mathrm{mult}_p(\pi) - 1$. Therefore the value of the intersection divisor $\mathrm{div}(\frac{\partial F}{\partial y})$ at p equals the value of the ramification divisor R_π at p. □

Theorem 1.27 **(Plücker's Formula)** *A smooth projective plane curve of degree d has genus $g = \frac{1}{2}(d-1)(d-2)$*

Proof Let X be a smooth projective plane curve defined by the homogenous polynomial $F(x, y, z) = 0$. Consider the holomorphic map $\pi : X \to \mathbb{P}^1$ by $\pi[x : y : z] = [x : z]$. π has degree d and $R_\pi = \operatorname{div}(\frac{\partial F}{\partial y})$ by Lemma 1.26. By Bezout's theorem, $\deg(\operatorname{div}(\frac{\partial F}{\partial y})) = \deg(X)(d-1) = d(d-1)$. By Hurwitz's formula, $2g-2 = \deg(\pi)(-2)+\sum_{p\in X}(\operatorname{mult}_p(\pi)-1)$. This is $2g-2 = -2d+d(d-1)$. Thus, $g = \frac{1}{2}(d-1)(d-2)$. $\square$

1.4 Sheaves and Cohomology

The use of sheaves gives a way to organize functions and forms satisfing local properties.

Definition 1.28 *Let X be a topological space. A presheaf of groups $\mathcal{F}$ on X is a collection of groups $\mathcal{F}(U)$, one for every open set U of X and a collection of group homomorphisms $\rho_V^U : \mathcal{F}(U) \to \mathcal{F}(V)$ whenever $V \subseteq U$ such that the followings hold.*

1. *$\mathcal{F}(\emptyset)$ is the trivial group with one element.*
2. *ρ_U^U =id on $\mathcal{F}(U)$*
3. *if $W \subseteq V \subseteq U$, then $\rho_W^U = \rho_W^V \circ \rho_V^U$*

Similarly, we can define a presheaf of rings if we take every $\mathcal{F}(U)$ as a ring and ρ_V^U as a ring homomorphism. The homomorphisms ρ_V^U are called restriction maps for the presheaf. Elements of $\mathcal{F}(U)$ are called sections of $\mathcal{F}$ over U. Elements of $\mathcal{F}(X)$ are called global sections of $\mathcal{F}$.

The definition of a presheaf requires that a property still holds after restriction. The sheaf axiom states the converse. A sheaf is a presheaf satisfying the sheaf axiom.

Definition 1.29 *Suppose $\mathcal{F}$ is a presheaf on X. For an open set U in X, let $\{U_i\}$ be an open cover of U. We say $\mathcal{F}$ satisfies the sheaf axiom for U and $\{U_i\}$ if whenever one has elements $s_i \in \mathcal{F}(U_i)$ such that $\rho^{U_i}_{U_i \cap U_j}(s_i) = \rho^{U_j}_{U_i \cap U_j}(s_j)$ for all i and j, then there exists a unique $s \in \mathcal{F}(U)$ such that $\rho^{U}_{U_i}(s) = s_i$ for all i. We say $\mathcal{F}$ is a sheaf if it satisfies the sheaf axiom for every open U and every open cover $\{U_i\}$ of U.*

Note that if $s, t \in \mathcal{F}(U)$ such that $\rho^{U}_{U_i}(s) = \rho^{U}_{U_i}(t)$ for all i, then $s = t$ by the sheaf axiom. If one has a presheaf of functions and forms defined by a local property, then the presheaf is a sheaf automatically.

We now give several examples of sheaves on a Riemann surface. One can easily verify that these examples are presheaves defined by local properties. One can also see that the global sections of these sheaves are spaces that we are familiar with.

Example Let X be a Riemann surface and $\mathcal{O}(U)$ be the ring of holomorphic functions on an open set U of X. With the usual restriction map $\mathcal{O}(U) \to \mathcal{O}(V)$ for $V \subseteq U$, this gives the sheaf $\mathcal{O}$ of holomorphic functions on X . The global sections of the sheaf $\mathcal{O}$ is $\mathcal{O}(X)$, the ring of holomorphic functions on X. Since every holomorphic function on a compact Riemann surface is constant, if X is a compact Riemann surface, global sections of the sheaf is $\mathcal{O}(X) = \mathbb{C}$. The sheaf $\mathcal{M}$ of meromorphic functions on X is defined similarly.

Example Let X be a Riemann surface and $\mathcal{O}^*(U)$ be the multiplicative group of all nowhere zero holomorphic functions on an open set U of X. This is a sheaf of groups $\mathcal{O}^*$. The global sections of the sheaf $\mathcal{O}^*$ is $\mathcal{O}^*(X)$, the multiplicative group of all nowhere zero holomorphic functions on X. If X is a compact Riemann surface, global sections $\mathcal{O}^*(X) = \mathbb{C}^*$. The sheaf $\mathcal{M}^*$ of nowhere zero meromorphic functions on X is defined similarly.

Example Let D be a divisor on a Riemann surface X. Let $\mathcal{O}[D](U)$ be the group of all meromorphic funtions $f : U \to \mathbb{C}$ on an open set U of X such that $\operatorname{ord}_p(f) \geq -D(p)$ for all $p \in U$. This is a sheaf of groups $\mathcal{O}[D]$. The global sections of the sheaf $\mathcal{O}[D]$ is $\mathcal{O}[D](X) = L(D)$.

Let X be a topological space and G be a group. If $U \subseteq X$ is an open subset, then a function $f : U \to G$ is locally constant if for every $p \in U$, there is a neighborhood $V \subseteq U$ of p such that f is constant on V. Since locally constant is a local property, this defines a sheaf on X. Denote this by $\underline{G}$, which is called a constant sheaf of X.

We can also define a skyscraper sheaf on X as follows. One assigns a group G_p for every $p \in X$. Then for every open set U of X, define $\mathcal{F}(U)$ as the subgroup of $\prod_{p \in U} G_p$ consisting sections with a discrete support, that is, the sections are direct products of $g_p \in G_p$ such that the set $\{p \in U : g_p \neq 0\}$ forms a discrete set. The restriction maps are the natural projections. This gives a skyscraper sheaf $\mathcal{F}$ on X.

Example Let X be a Riemann surface. We assign the group of integers $\mathbb{Z}$ at every point of X. For every open U of X, we define a group of functions from U to $\mathbb{Z}$ which are discretely supported. This gives a skyscraper sheaf $\mathcal{D}iv_X$. The global sections of the sheaf $\mathcal{D}iv_X$ is $\operatorname{Div}(X)$.

Definition 1.30 *Let $\mathcal{F}$ and $\mathcal{G}$ be two sheaves on a topological space X. A sheaf map from $\mathcal{F}$ to $\mathcal{G}$ is a collection of homomorphisms $\phi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ for every open set U of X, which commute with the restriction maps, that is $\rho^U_V \circ \phi_U = \phi_V \circ \rho^U_V$ whenever $V \subseteq U$ is an open subset. The sheaf map is denoted by ϕ.*

Definition 1.31 *Suppose $\phi : \mathcal{F} \to \mathcal{G}$ is a sheaf map. For each open set U of X, define $\mathcal{K}(U)$ to be the kernel of the group homomorphism $\phi_U : \mathcal{F}(U) \to \mathcal{G}(U)$, that is $\mathcal{K}(U) = ker(\phi_U)$. Then $\mathcal{K}$ is a subsheaf of $\mathcal{F}$, $\mathcal{K} \subset \mathcal{F}$. We say $\mathcal{K}$ is the kernel sheaf of ϕ.*

To check the above is well-defined, we need to show that $\mathcal{K}$ is a sheaf, that is a presheaf satisfying the sheaf axiom. For every open subset $V \subseteq U$, suppose $s \in \mathcal{K}(U)$, $\phi_V(\rho_V^U(s)) = \rho_V^U(\phi_U(s)) = \rho_V^U(0) = 0$. Hence $\mathcal{K}$ is a presheaf. Then we check the sheaf axiom for $\mathcal{K}$. Let $\{U_i\}$ be an open cover of U. Suppose $s_i \in \mathcal{K}(U_i)$ such that $\rho_{U_i \cap U_j}^{U_i}(s_i) = \rho_{U_i \cap U_j}^{U_j}(s_j)$ for all i. Since $\mathcal{F}$ is a sheaf, there exists a unique $s \in \mathcal{F}(U)$ such that $\rho_{U_i}^U(s) = s_i$ for all i. It remains to show that s is in $\mathcal{K}(U)$, that is $\phi_U(s) = 0$ in $\mathcal{G}(U)$. $\rho_{U_i}^U(\phi_U(s)) = \phi_{U_i}(\rho_{U_i}^U(s)) = \phi_{U_i}(s_i) = 0$ for each i. Since G is a sheaf, it gives $\phi_U(s) = 0$. Hence $\mathcal{K}$ is a sheaf.

Definition 1.32 *Let $\phi : \mathcal{F} \to \mathcal{G}$ be a sheaf map. We say ϕ is one-to-one if for all $p \in X$ and all open sets U containing p, there exists an open subset $V \subseteq U$ containing p such that ϕ_V is one-to-one. We say ϕ is onto if for all $p \in X$ and all open set U containing p, there exists an open subset $V \subseteq U$ containing p such that ϕ_V is onto.*

Proposition 1.33 *The followings are equivalent for a sheaf map $\phi : \mathcal{F} \to \mathcal{G}$*

1. *ϕ is one-to-one*

2. *ϕ_U is one-to-one for every open set U of X*

3. *kernel sheaf for ϕ is identically zero sheaf.*

Proof Clearly *2⇒1* and *2⇔3*. It suffices to show *1⇒2*. For any open set U of X and let $s \in \mathcal{F}(U)$ such that $\phi_U(s) = 0$ in $\mathcal{G}(U)$, we want to show that $s = 0$ in $\mathcal{F}(U)$. Since ϕ is one-to-one, for all $p \in U$, there exists an open subset $V_p \subseteq U$ containing p such that ϕ_{V_p} is one-to-one. $\phi_{V_p}(\rho_{V_p}^U(s)) = \rho_{V_p}^U(\phi_U(s)) = \rho_{V_p}^U(0) = 0$.

Since ϕ_{V_p} is one-to-one, $\rho^U_{V_p}(s) = 0$. Since $\{V_p\}_{p \in U}$ is an open cover of U and $\mathcal{F}$ is a sheaf, $s = 0$ in $\mathcal{F}(U)$. $\square$

Note that analogy of the above lemma is not true for onto maps of sheaves. One can see this by the example of the sheaf map $\exp(2\pi i -) : \mathcal{O} \to \mathcal{O}^*$ where $X = \mathbb{C}^*$. That is, for every open set U of X and every $f \in \mathcal{O}(U)$, $\exp(2\pi i -)(f) = \exp(2\pi i f) \in \mathcal{O}^*(U)$. Consider $\frac{1}{z} \in \mathcal{O}^*(X)$. There is no holomorphic function f such that $\exp(2\pi i f) = \frac{1}{z}$, thus $\exp(2\pi i -) : X \to \mathcal{O}^*(X)$ is not onto. However, for all $p \in \mathbb{C}^*$, there exists a branch of $\ln(z)$ defined near p. $f(z) = -\frac{1}{2\pi i}\ln(z)$ then satisfies $\exp(2\pi i f) = \frac{1}{z}$ on a neighborhood of p. Hence the sheaf map is onto.

Definition 1.34 *A sequence of sheaf maps $0 \to \mathcal{K} \to \mathcal{F} \xrightarrow{\phi} \mathcal{G} \to 0$ is a short exact sequence of sheaves if the sheaf map ϕ is onto and the sheaf $\mathcal{K}$ is the kernel sheaf of ϕ.*

While the use of sheaves is a way to solve local problems, cohomology is used to solve global problems. We now develop the notions of cohomology.

Definition 1.35 *Let $\mathcal{F}$ be a sheaf of topological space X with an open cover $\mathcal{U} = \{U_i\}$. A Cěch n-cochain for the sheaf $\mathcal{F}$ over the open cover $\mathcal{U}$ is a collection of sections of $\mathcal{F}$, one for each $U_{i_0} \cap U_{i_1} \cap \ldots \cap U_{i_n}$. Denote $\check{C}^n(\mathcal{U}, \mathcal{F})$ as the space of Cěch n-cochains for $\mathcal{F}$ over $\mathcal{U}$, $\check{C}^n(\mathcal{U}, \mathcal{F}) = \prod_{(i_0, i_1, \ldots, i_n)} \mathcal{F}(U_{i_0, i_1, \ldots, i_n})$ where $U_{i_0, i_1, \ldots, i_n} = U_{i_0} \cap U_{i_1} \cap \ldots \cap U_{i_n}$.*

A Cěch 0-cochain is a collection of $f_i \in \mathcal{F}(U_i)$, one for each i. A Cěch 1-cochain is a collection of $f_{ij} \in \mathcal{F}(U_i \cap U_j)$, one for each pair of i and j. Denote a Cěch n-cochain by $(f_{i_0, i_1, \ldots, i_n})$. Note that a sheaf map $\phi : \mathcal{F} \to \mathcal{G}$ induces $\phi : \check{C}^n(\mathcal{U}, \mathcal{F}) \to \check{C}^n(\mathcal{U}, \mathcal{G})$ by sending $(f_{i_0, i_1, \ldots, i_n})$ to $(\phi_{U_{i_0} \cap U_{i_1} \cap \ldots \cap U_{i_n}}(f_{i_0, i_1, \ldots, i_n}))$.

Definition 1.36 *Define the coboundary operator* $\delta : \check{C}^n(\mathcal{U}, \mathcal{F}) \to \check{C}^{n+1}(\mathcal{U}, \mathcal{F})$ *by* $\delta(f_{i_0,i_1,\ldots,i_n}) = (g_{i_0,i_1,\ldots,i_{n+1}})$ *where* $g_{i_0,i_1,\ldots,i_{n+1}} = \sum_{k=0}^{n+1}(-1)^k \rho(f_{i_0,i_1,\ldots,\widehat{i_k},\ldots,i_{n+1}})$, *and* ρ *denotes the restriction map for the sheaf* $\mathcal{F}$ *corresponding to the subset* $U_{i_0,i_1,\ldots,i_{n+1}} \subseteq U_{i_0,i_1,\ldots,\widehat{i_k},\ldots,i_{n+1}}$

At 0-level, δ sends a 0-cochain (f_i) to a 1-cochain (g_{ij}) where $g_{ij} = f_j - f_i$ on U_{ij}. At 1-level, δ sends a 1-cochain (f_{ij}) to a 2-cochain (g_{ijk}) where $g_{ijk} = f_{jk} - f_{ik} + f_{ij}$ on U_{ijk}.

Definition 1.37 *A n-cocycle is a n-cochain c with* $\delta c = 0$. *We denote the space of n-cocycles by* $\check{Z}^n(\mathcal{U}, \mathcal{F})$. *A n-coboundary is a n-cochain which is an image of* δ. *We denote the space of n-coboundaries by* $\check{B}^n(\mathcal{U}, \mathcal{F})$.

At 0-level, $\delta \circ \delta(f_i) = \delta(f_j - f_i) = \delta(g_{ij})$ where $g_{ij} = f_j - f_i$. $\delta(g_{ij}) = g_{jk} - g_{ik} + g_{ij} = f_k - f_j - f_k + f_i + f_j - f_i = 0$, thus $\delta \circ \delta(f_i) = 0$. In general, $\delta \circ \delta = 0$ also true for any level. At n-level, $\delta \circ \delta(f_{i_0,\ldots,i_n}) = \delta(g_{i_0,\ldots,i_{n+1}})$ where $g_{i_0,\ldots,i_{n+1}} = \sum_{k=0}^{n+1}(-1)^k \rho(f_{i_0,\ldots,\widehat{i_k},\ldots,i_{n+1}})$. $\delta(g_{i_0,\ldots,i_{n+1}}) = \sum_{j=0}^{n+2}(-1)^j \rho(g_{i_0,\ldots,\widehat{i_j},\ldots,i_{n+2}})$ $= \sum_{j=0}^{n+2}(-1)^j \sum_{k<j}(-1)^{j+k} f_{i_0,\ldots,\widehat{i_k},\ldots,\widehat{i_j},\ldots,i_{n+2}} + \sum_{j=0}^{n+2}(-1)^j \sum_{k\geq j}(-1)^{j+k+1}$ $f_{i_0,\ldots,\widehat{i_j},\ldots,\widehat{i_k},\ldots,i_{n+2}} = 0$. Since $\delta \circ \delta = 0$, we have a Cěch cochain complex $0 \to \check{C}^0(\mathcal{U}, \mathcal{G}) \xrightarrow{\delta} \check{C}^1(\mathcal{U}, \mathcal{G}) \xrightarrow{\delta} \check{C}^2(\mathcal{U}, \mathcal{G}) \xrightarrow{\delta} \cdots$. Also, every n-coboundary is a n-cocycle, thus $\check{B}^n(\mathcal{U}, \mathcal{F}) \subseteq \check{Z}^n(\mathcal{U}, \mathcal{F})$.

Definition 1.38 *The* n^{th} *cohomology group* $\check{H}^n(\mathcal{U}, \mathcal{F})$ *of* $\mathcal{F}$ *with respect to the open cover* $\mathcal{U}$ *is the quotient group* $\check{H}^n(\mathcal{U}, \mathcal{F}) = \check{Z}^n(\mathcal{U}, \mathcal{F})/\check{B}^n(\mathcal{U}, \mathcal{F})$.

Lemma 1.39 *For any open cover* $\mathcal{U}$, *the* 0^{th} *cohomology group of a sheaf* $\mathcal{F}$ *is isomorphic to the group of global sections of* $\mathcal{F}$, *that is,* $\check{H}^0(\mathcal{U}, \mathcal{F}) = \mathcal{F}(X)$.

Proof Since there is no 0-coboundary, $\check{H}^0(\mathcal{U}, \mathcal{F}) = \check{Z}^0(\mathcal{U}, \mathcal{F})$. Define $\alpha : \mathcal{F}(X) \to \check{C}^0(\mathcal{U}, \mathcal{F})$ by sending global section f to 0-cochain (f_i) where $f_i = \rho^X_{U_i}(f)$. $\delta(f_i) = (g_{ij})$, where $g_{ij} = f_i - f_j = 0$ since f_i, f_j are restrictions of f. Thus, α maps $\mathcal{F}(X)$ to $\check{Z}^0(\mathcal{U}, \mathcal{F}) = \check{H}^0(\mathcal{U}, \mathcal{F})$. α is then one-to-one and onto by the sheaf axiom. □

A sheaf map $\phi : \mathcal{F} \to \mathcal{G}$ induces $\phi : \check{C}^n(\mathcal{U}, \mathcal{F}) \to \check{C}^n(\mathcal{U}, \mathcal{G})$. Since the coboundary map δ commutes ϕ, ϕ sends cocycles to cocycles and coboundaries to coboundaries. Hence ϕ induces $\phi_* : \check{H}^n(\mathcal{U}, \mathcal{F}) \to \check{H}^n(\mathcal{U}, \mathcal{G})$.

Definition 1.40 *Let $\mathcal{U} = \{U_i\}_{i \in I}$ and $\mathcal{V} = \{V_j\}_{j \in J}$ be two open covers of X. $\mathcal{V}$ is a refinement of $\mathcal{U}$, denoted as $\mathcal{V} \prec \mathcal{U}$, if for every open set V_j of $\mathcal{V}$, there is an open set U_i of $\mathcal{U}$ such that $V_j \subseteq U_i$.*

Let $\mathcal{V} = \{V_j\}_{j \in J}$ be a refinement of $\mathcal{U} = \{U_i\}_{i \in I}$. Let r be the refining map between two index sets, $r : J \to I$. Then r induces a map from $\check{C}^n(\mathcal{U}, \mathcal{F})$ to $\check{C}^n(\mathcal{V}, \mathcal{F})$ by sending $(f_{i_0, \dots, i_n})$ to $(g_{j_0, \dots, j_n})$ where $g_{j_0, \dots, j_n} = f_{r(j_0), \dots, r(j_n)}|_{V_{j_0, j_1, \dots, j_n}}$. This map sends cocycles to cocycles and coboundaries to coboundaries, and thus induces a homomorphism $H^{\mathcal{U}}_{\mathcal{V}} : \check{H}^n(\mathcal{U}, \mathcal{F}) \to \check{H}^n(\mathcal{V}, \mathcal{F})$. Note that $H^{\mathcal{U}}_{\mathcal{V}}$ is independent of the choice of the refining map r and is one-to-one.

If $\mathcal{W} \prec \mathcal{U} \prec \mathcal{V}$, then $H^{\mathcal{V}}_{\mathcal{W}} \circ H^{\mathcal{U}}_{\mathcal{V}} = H^{\mathcal{U}}_{\mathcal{W}}$. One can define an equivalence relation on the disjoint union of $\check{H}^n(\mathcal{U}, \mathcal{F})$, where $\mathcal{U}$ runs through all open covers of X. For $\xi \in \check{H}^n(\mathcal{U}, \mathcal{F})$ and $\eta \in \check{H}^n(\mathcal{V}, \mathcal{F})$, we say $\xi \sim \eta$ if there exists an open cover $\mathcal{W}$ with $\mathcal{W} \prec \mathcal{U}$ and $\mathcal{W} \prec \mathcal{V}$ such that $H^{\mathcal{U}}_{\mathcal{W}}(\xi) = H^{\mathcal{V}}_{\mathcal{W}}(\eta)$. Denote the set of equivalence classes as $\lim_{\overrightarrow{\mathcal{U}}} \check{H}^n(\mathcal{U}, \mathcal{F})$.

Suppose $x, y \in \lim_{\overrightarrow{\mathcal{U}}} \check{H}^n(\mathcal{U}, \mathcal{F})$ are represented by $\xi \in \check{H}^n(\mathcal{U}, \mathcal{F})$ and $\eta \in \check{H}^n(\mathcal{V}, \mathcal{F})$ respectively. Let $\mathcal{W}$ be the common refinement of $\mathcal{U}$ and $\mathcal{V}$. Then $x + y \in \check{H}^n(X, \mathcal{F})$ is defined to be the equivalence class of $H^{\mathcal{U}}_{\mathcal{W}}(\xi) + H^{\mathcal{V}}_{\mathcal{W}}(\eta)$. This makes $\lim_{\overrightarrow{\mathcal{U}}} \check{H}^n(\mathcal{U}, \mathcal{F})$ into an abelian group.

Definition 1.41 *Fix a sheaf $\mathcal{F}$ on a Riemann surafce X and an integer $n \geq 0$. The n^{th} Cĕch Cohomology group of $\mathcal{F}$ on X is the group $\check{H}^n(X, \mathcal{F}) = \lim_{\overrightarrow{\mathcal{U}}} \check{H}^n(\mathcal{U}, \mathcal{F})$.*

Suppose $\phi : \mathcal{F} \to \mathcal{G}$ is an onto map of sheaves, and $\mathcal{K}$ is the kernel sheaf for ϕ. Define the connecting homomorphism $\Delta : \check{H}^0(X, \mathcal{G}) \to \check{H}^1(X, \mathcal{K})$ as follows. For

$g \in \mathcal{G}(X) \cong \check{H}^0(X, \mathcal{G})$ and $p \in X$, since ϕ is onto, there exists a neighborhood U_p of p such that $g = \phi(f_p)$ on U_p for some $f_p \in \mathcal{F}(U_p)$. Then $\mathcal{U} = \{U_p\}_{p \in X}$ is an open cover of X. Let $f_{pq} = f_p - f_q \in \mathcal{F}(U_p \cap U_q)$. (f_{pq}) is a 1-cocycle for the sheaf $\mathcal{F}$. Also, $\phi(f_{pq}) = \phi(f_p) - \phi(f_q) = g - g = 0$, thus (f_{pq}) is also a 1-cocycle for the kernel sheaf $\mathcal{K}$. (f_{pq}) represents a class in $\check{H}^1(\mathcal{U}, \mathcal{K})$. $\Delta(g)$ is defined to be the equivalence class of (f_{pq}) in $\check{H}^1(X, \mathcal{K})$. Note that $\Delta(g)$ is independent of choice of the open cover $\mathcal{U}$ and the preimages f_p.

Lemma 1.42 *Let $\phi : \mathcal{F} \to \mathcal{G}$ be an onto sheaf map of X. Suppose $g \in \mathcal{G}(X)$ is a global section of $\mathcal{G}$. Then there exists a global section $s \in \mathcal{F}(X)$ of $\mathcal{F}$ such that $\phi(s) = g$ if and only if $\Delta(g) = 0$.*

Proof Suppose $\phi(s) = g$ for some $s \in \mathcal{F}(X)$. For each $p \in X$, choose $U_p = X$ with $f_p = s$ on X. Then $f_{pq} = f_p - f_q = 0$ for all p, q, $(f_{pq}) = 0$ as a 1-cocycle. Hence $\Delta(g) = 0$.

Conversely, suppose $\Delta(g) = 0$ in $\check{H}^1(X, \mathcal{K})$. Since ϕ is onto, write $\phi((f_p)) = g$ on U_p for some $f_p \in \mathcal{F}(U_p)$. Since $\Delta(g) = 0$, (f_{pq}) is a coboundary. Write $f_{pq} = h_q - h_p$ for some 0-cochain (h_p) for $\mathcal{K}$. Let $s_p = f_p - h_p$ on U_p. On $U_p \cap U_q$, $s_p - s_q = (h_q - h_p) - (f_q - f_p) = 0$. By the sheaf axiom, $\{s_p\}$ patch to the global section $s \in \mathcal{F}(X)$. On each U_p, $g = \phi((f_p)) = \phi((f_p - h_p)) = \phi((s_p)) = \phi(s)$. By the sheaf axiom, $g = \phi(s)$ on X. □

By Lemma 1.42, we can see that $\mathcal{F}(X) \xrightarrow{\phi} \mathcal{G}(X) \xrightarrow{\Delta} \check{H}^1(X, \mathcal{K})$ is an exact sequence of groups. In fact, it is a part of the long exact sequence of cohomology groups.

Proposition 1.43 *Let $\phi : \mathcal{F} \to \mathcal{G}$ be an onto map of sheaves with the kernel sheaf $\mathcal{K}$. Then $0 \to \mathcal{K}(X) \xrightarrow{inc} \mathcal{F}(X) \xrightarrow{\phi} \mathcal{G}(X) \xrightarrow{\Delta} \check{H}^1(X, \mathcal{K}) \xrightarrow{inc_*} \check{H}^1(X, \mathcal{F}) \xrightarrow{\phi_*} \check{H}^1(X, \mathcal{G})$ is exact at every step.*

Proof Exactness at $\mathcal{K}(X)$ and $\mathcal{F}(X)$ follows from definition. Exactness at $\mathcal{G}(X)$ follows by Lemma 1.42. It remains to show that it is exact at $\check{H}^1(X,\mathcal{K})$ and $\check{H}^1(X,\mathcal{F})$.

Suppose $g \in \mathcal{G}(X)$. Choose an open cover $\{U_i\}$ and $f_i \in \mathcal{F}(U_i)$ such that $\phi_{U_i}(f_i) = g|_{U_i}$. $\Delta(g) = (f_i - f_j)$ as a 1-cocycle for the sheaf $\mathcal{K}$, which is also a 1-coboundary for the sheaf $\mathcal{F}$. So, $im(\Delta) \subseteq ker(inc_*)$. Suppose (h_{ij}) a 1-cocycle for the sheaf $\mathcal{K}$ which represents a class in $ker(inc_*)$. Then (h_{ij}) is a coboundary for the sheaf $\mathcal{F}$, that is, there exists 0-cochain (f_i) such that $h_{ij} = f_j - f_i$ on $U_i \cap U_j$. Consider 0-cochain (g_i) for $\mathcal{G}$, where $g_i = \phi(f_i)$. $g_i - g_j = \phi(f_i - f_j) = \phi(h_{ij}) = 0$ on $U_i \cap U_j$. By the sheaf axiom, there exists $g \in \mathcal{G}(X)$ such that $g|_{U_i} = g_i$ for all i. Hence $\Delta(g) = (h_{ij})$. Therefore $ker(inc_*) \subseteq im(\Delta)$. This prove the exactness at $\check{H}^1(X,\mathcal{K})$

Clearly, $\phi_* \circ inc_* = 0$. So, $im(inc_*) \subseteq ker(\phi_*)$. Suppose (f_{ij}) is a 1-cocycle for $\mathcal{F}$, represents a class $c \in ker(\phi_*)$, that is $\phi_*(c) = 0$ in $\check{H}^1(X,\mathcal{G})$. Then $(\phi(f_{ij}))$ is a coboundary for $\mathcal{G}$, that is, there exists a 0-cochain (g_i) such that $\phi(f_{ij}) = g_j - g_i$. Since ϕ is onto, there exist k_i, k_j such that $g_i = \phi(k_i)$, $g_j = \phi(k_j)$. Let $h_{ij} = f_{ij} - k_j + k_i \in \mathcal{F}(U_i \cap U_j)$ which is a 1-cocycle. $\phi(h_{ij}) = \phi(f_{ij}) - g_j + g_i = 0$, thus (h_{ij}) is a 1-cocycle for $\mathcal{K}$. And $(h_{ij}) - (f_{ij}) = \delta(k_i)$, thus (h_{ij}) and (f_{ij}) represent the same class in $\check{H}^1(X,\mathcal{F})$. $(f_{ij}) = inc_*(h_{ij})$. So, $ker(\phi_*) \subseteq im(inc_*)$. This prove the exactness at $\check{H}^1(X,\mathcal{F})$. $\square$

An open cover of a topological space is locally finite if every point has a neighborhood which interests only finitely many of open sets in the open cover. A space is paracompact if it is Hausdorff and every open cover has a locally finite refinement. We have the following long exact sequence of cohomology groups for a paracompact space.

Theorem 1.44 *Let X be a paracompact space and $0 \to \mathcal{K} \to \mathcal{F} \to \mathcal{G} \to 0$ be a short exact sequence of sheaves on X. Then there exist connecting homomorphisms $\Delta : \check{H}^n(X,\mathcal{G}) \to \check{H}^{n+1}(X,\mathcal{K})$ for every $n \geq 0$ such that the sequence of* *cohomology groups* $0 \to \check{H}^0(X,\mathcal{K}) \overset{inc_*}{\to} \check{H}^0(X,\mathcal{F}) \overset{\phi_*}{\to} \check{H}^0(X,\mathcal{G}) \overset{\Delta}{\to} \check{H}^1(X,\mathcal{K}) \overset{inc_*}{\to} \check{H}^1(X,\mathcal{F}) \overset{\phi_*}{\to} \check{H}^1(X,\mathcal{G}) \overset{\Delta}{\to} \cdots$ *is exact.*

Chapter 2

The Riemann-Roch Theorem and Algebraic Curves

2.1 Finiteness Theorem

This section is based on the textbook [1]. We are going to prove the existence of non-constant meromorphic functions on every compact Riemann surface.

Let D be an open set in $\mathbb{C}$ and $f \in \mathcal{O}(D)$. Define $\|f\|_{L^2(D)} = (\iint_D |f(x+iy)|^2 dxdy)^{\frac{1}{2}}$. Denote $L^2(D, \mathcal{O})$ be the vector space of all holomorphic functions on D which $\|f\|_{L^2(D)} < \infty$. We can then define an inner product of $f, g \in L^2(D, \mathcal{O})$ by $\langle f, g \rangle = \iint_D f\bar{g} dxdy$. $\iint_D f\bar{g} dxdy < \infty$ because $|f\bar{g}| \leq \dfrac{1}{2}(|f|^2 + |g|^2)$.

Let $B = B(a, r) = \{z \in \mathbb{C} : |z - a| < r\}$. Consider the orthogonal system $\phi_n(z) = (z-a)^n, n \in \mathbb{N}$ for $L^2(B, \mathcal{O})$. Then $\|\phi_n\|_{L^2(B)} = \dfrac{\sqrt{\pi} r^{n+1}}{\sqrt{n+1}}$. Let $f \in L^2(B, \mathcal{O})$, write $f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$. Then $\|f\|^2_{L^2(B)} = \sum_{n=0}^{\infty} |c_n|^2 \dfrac{\pi r^{2n+1}}{n+1}$.

Theorem 2.1 *Let D be an open subset of $\mathbb{C}$, $r > 0$ and $D_r = \{z \in \mathbb{C} : B(z, r) \subset D\}$. Then for all $f \in L^2(D, \mathcal{O})$, $\|f\|_{D_r} \leq \dfrac{1}{\sqrt{\pi} r} \|f\|_{L^2(D)}$, where $\|f\|_{D_r} = \sup\{|f(a)| : a \in D_r\}$.*

Proof Let $a \in D_r$. Write $f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$ as the Taylor series of f

about a. $\|f\|^2_{L^2(B)} = \sum_{n=0}^{\infty} |c_n|^2 \dfrac{\pi r^{2n+1}}{n+1}$ implies $|f(a)| = |c_0| \leq \dfrac{1}{\sqrt{\pi r}} \|f\|_{L^2(B)}$, also $\dfrac{1}{\sqrt{\pi r}} \|f\|_{L^2(B)} \leq \dfrac{1}{\sqrt{\pi r}} \|f\|_{L^2(D)}$. Hence, $\sup\{|f(a)| : a \in D_r\} \leq \dfrac{1}{\sqrt{\pi r}} \|f\|_{L^2(D)}$. □

Hence, $L^2(D, \mathcal{O})$ is a Hilbert space.

D' is relatively compact on D if $\overline{D'}$ is compact in D. Denote this relation by $D' \Subset D$.

Lemma 2.2 **(Generalization of Schwarz's Lemma)** *Let $D' \Subset D$ be open subsets of $\mathbb{C}$. Then for all $\varepsilon > 0$, there exists a closed vector subspace $A \subset L^2(D, \mathcal{O})$ of finite codimension such that $\|f\|_{L^2(D')} \leq \varepsilon \|f\|_{L^2(D)}$ for all $f \in A$.*

Proof Since $\overline{D'}$ is compact and lies in D, there exists $r > 0$ and finitely many $a_1, \ldots, a_k$ in D such that $B(a_i, r) \subset D$ and $D' \subset \bigcup_{i=1}^{k} B(a_i, \dfrac{r}{2})$. Choose n large enough that $2^{-n-1}k \leq \varepsilon$. Let A be the set of f in $L^2(D, \mathcal{O})$ where f vanishes at all a_i at least to order n. Then A is a closed vector subspace of $L^2(D, \mathcal{O})$ of codimension $\leq kn$. Let $f \in A$, write $f(z) = \sum_{i=n}^{\infty} c_i (z - a_i)^i$. $\|f\|^2_{L^2(B(a_i, \frac{r}{2}))} = \sum_{i=n}^{\infty} |c_i|^2 \dfrac{\pi (\frac{r}{2})^{2i+2}}{i+1}$ implies $\|f\|_{L^2(B(a_i, \frac{r}{2}))} \leq \dfrac{1}{2^{n+1}} (\sum_{i=n}^{\infty} |c_i|^2 \dfrac{\pi r^{2i+1}}{i+1})^{\frac{1}{2}} = \dfrac{1}{2^{n+1}} \|f\|_{L^2(B(a_i, r))}$. Also, $\|f\|_{L^2(D')} \leq \sum_{i=1}^{k} \|f\|_{L^2(B(a_i, \frac{r}{2}))}$ and $\|f\|_{L^2(B(a_i, r))} \leq \|f\|_{L^2(D)}$. Hence, $\|f\|_{L^2(D')} \leq \sum_{i=1}^{k} (\dfrac{1}{2^{n+1}} \|f\|_{L^2(B(a_i, r))}) \leq \dfrac{k}{2^{n+1}} \|f\|_{L^2(D)} \leq \varepsilon \|f\|_{L^2(D)}$. □

Let X be a Riemann surface. Choose a finite family of charts on X, $(U_i^*)_{1 \leq i \leq n}$ with local coordinate z_i on U_i^* and $z_i(U_i^*) \subset \mathbb{C}$ is a disk. Let U_i be an open set in U_i^*. and $\mathcal{U} = (U_i)_{1 \leq i \leq n}$. We introduce L^2-norms on cochain groups $\check{C}^0(\mathcal{U}, \mathcal{O})$ and $\check{C}^1(\mathcal{U}, \mathcal{O})$ as follows.

1. for all $\eta = (f_i) \in \check{C}^0(\mathcal{U}, \mathcal{O})$, define $\|\eta\|^2_{L^2(\mathcal{U})} = \sum_i \|f_i \circ z_i^{-1}\|^2_{L^2(z_i(U_i))}$

2. for all $\xi = (f_{ij}) \in \check{C}^1(\mathcal{U}, \mathcal{O})$, define $\|\xi\|^2_{L^2(\mathcal{U})} = \sum_{i,j} \|f_{ij} \circ z_i^{-1}\|^2_{L^2(z_i(U_i \cap U_j))}$

Denote $\check{C}^0_{L^2}(\mathcal{U}, \mathcal{O}) \subset \check{C}^0(\mathcal{U}, \mathcal{O})$ and $\check{C}^1_{L^2}(\mathcal{U}, \mathcal{O}) \subset \check{C}^1(\mathcal{U}, \mathcal{O})$ as sets of cochains having finite norms. They are Hilbert spaces. Denote $\check{Z}^1_{L^2}(\mathcal{U}, \mathcal{O})$ as set of cocycles in $\check{C}^1_{L^2}(\mathcal{U}, \mathcal{O})$. It forms a closed vector subspace in $\check{C}^1_{L^2}(\mathcal{U}, \mathcal{O})$, hence is a Hilbert space.

Let $V_i \Subset U_i$ for $1 \leq i \leq n$ and $\mathcal{V} = (V_i)_{1\leq i\leq n}$. Denote the above relation by $\mathcal{V} \ll \mathcal{U}$. Also denote $|\mathcal{U}| = U_1 \cup \ldots \cup U_n$.

Theorem 2.3 *Suppose X is a Riemann surface. Then $\check{H}^1(X, \mathcal{E})$, $\check{H}^1(X, \mathcal{E}^{(1)})$, $\check{H}^1(X, \mathcal{E}^{(1,0)})$, $\check{H}^1(X, \mathcal{E}^{(0,1)})$ all vanish.*

Theorem 2.4 **(Dolbeault Lemma)** *Suppose $X = \{z \in \mathbb{C} : |z| < r\}$, $0 < r \leq \infty$, and $g \in \mathcal{E}(X)$ where $\mathcal{E}(X)$ is the set of C^∞ functions on X. Then there exists $f \in \mathcal{E}(X)$ such that $\dfrac{\partial f}{\partial \bar{z}} = g$.*

By Theorem 2.4, one can prove the following.

Theorem 2.5 *Suppose $X = \{z \in \mathbb{C} : |z| < r\}$, $0 < r \leq \infty$. Then $\check{H}^1(X, \mathcal{O}) = 0$.*

Lemma 2.6 *Let X be a Riemann surface and $(U_i^*)_{1\leq i\leq n}$ be a finite family of charts on X with local coordinate z_i on U_i^* and $z_i(U_i^*) \subset \mathbb{C}$ is a disc. Let $\mathcal{W} \ll \mathcal{V} \ll \mathcal{U} \ll \mathcal{U}^*$. Then there exists a constant $c > 0$ such that for all $\xi \in \check{Z}^1_{L^2}(\mathcal{V}, \mathcal{O})$, there exist $\zeta \in \check{Z}^1_{L^2}(\mathcal{U}, \mathcal{O})$ and $\eta \in \check{C}^0_{L^2}(\mathcal{W}, \mathcal{O})$ such that $\zeta = \xi + \delta\eta$ on $\mathcal{W}$, and* $\max(\|\zeta\|_{L^2(\mathcal{U})}, \|\eta\|_{L^2(\mathcal{W})}) \leq c\|\xi\|_{L^2(\mathcal{V})}$.

Proof Let $\xi = (f_{ij}) \in \check{Z}^1_{L^2}(\mathcal{V}, \mathcal{O})$. By Theorem 2.3, $\check{H}^1(\mathcal{V}, \mathcal{E}) = 0$, thus (f_{ij}) is a coboundary. There exists $(g_i) \in \check{C}^0(\mathcal{V}, \mathcal{E})$ such that $f_{ij} = g_j - g_i$ on $V_i \cap V_j$ for all i and j. Since $\dfrac{\partial f_{ij}}{\partial \bar{z}} = 0$, $\dfrac{\partial g_i}{\partial \bar{z}} = \dfrac{\partial g_j}{\partial \bar{z}}$ on $V_i \cap V_j$. Hence, there exists $\omega \in \mathcal{E}^{(0,1)}(|\mathcal{V}|)$ such that $\omega|_{V_i} = \dfrac{\partial g_i}{\partial \bar{z}} d\bar{z}$ for all i.

Since $|\mathcal{W}| \Subset |\mathcal{V}|$, by partition of unity, there exists $\psi \in \mathcal{E}(X)$ such that *supp*$(\psi) \subset |\mathcal{V}|$ and $\psi|_{|\mathcal{V}|} = 1$. By the Dolbeault Lemma, there exists $h_i \in \mathcal{E}(U_i^*)$ such that $\dfrac{\partial h_i}{\partial \bar{z}} d\bar{z} = \psi\omega$ on U_i^* for all i. Since $\dfrac{\partial h_i}{\partial \bar{z}} = \dfrac{\partial h_j}{\partial \bar{z}}$ on $U_i^* \cap U_j^*$, $F_{ij} = h_j - h_i \in \mathcal{O}(U_i^* \cap U_j^*)$. Define $\zeta = (F_{ij})|_{\mathcal{U}}$. Since $\mathcal{U} \ll \mathcal{U}^*$, $\zeta \in \check{Z}^1_{L^2}(\mathcal{U}, \mathcal{O})$. $\dfrac{\partial h_i}{\partial \bar{z}} d\bar{z} = \psi\omega = \omega = \dfrac{\partial g_i}{\partial \bar{z}} d\bar{z}$ on W_i. This implies $h_i - g_i$ is holomorphic on W_i.

Define $\eta = (h_i - g_i)|_{\mathcal{W}}$. Since $h_i - g_i$ bounded on W_i, $\eta \in \check{C}^1_{L^2}(\mathcal{W}, \mathcal{O})$. Finally, $F_{ij} - f_{ij} = (h_j - h_i) - (g_j - g_i) = (h_j - g_j) - (h_i - g_i)$ on $W_i \cap W_j$. This proves $\zeta = \xi + \delta\eta$ on $\mathcal{W}$.

Consider the Hilbert space $H = \check{Z}^1_{L^2}(\mathcal{U}, \mathcal{O}) \times \check{Z}^1_{L^2}(\mathcal{V}, \mathcal{O}) \times \check{Z}^1_{L^2}(\mathcal{W}, \mathcal{O})$ with norm $\|(\zeta, \xi, \eta)\|_H = (\|\zeta\|^2_{L^2(\mathcal{U})} + \|\xi\|^2_{L^2(\mathcal{V})} + \|\eta\|^2_{L^2(\mathcal{W})})^{\frac{1}{2}}$. Let $L \subset H$ be the subspace $L = \{(\zeta, \xi, \eta) \in H : \zeta = \xi + \delta\eta \text{ on } \mathcal{W}\}$. Since L is closed in H, L is a Hilbert space. The first part of this proof shows that the continuous linear map $\pi : L \longrightarrow \check{Z}^1_{L^2}(\mathcal{V}, \mathcal{O})$ by $\pi(\zeta, \xi, \eta) = \xi$ is onto. By the Banach theorem, π is open, and there exists $c > 0$ such that for all $\xi \in \check{Z}^1_{L^2}(\mathcal{V}, \mathcal{O})$, there exists $x = (\zeta, \xi, \eta) \in L$ such that $\|x\|_H \leq c\|\xi\|^2_{L^2(\mathcal{V})}$. □

Lemma 2.7 *Let X be a Riemann surface and $(U_i^*)_{1 \leq i \leq n}$ be a finite family of charts on X with local coordinate z_i on U_i^* and $z_i(U_i^*) \subset \mathbb{C}$ is a disc. Let $\mathcal{W} \ll \mathcal{V} \ll \mathcal{U} \ll \mathcal{U}^*$. Then there exists a finite dimensional vector subspace $S \subset \check{Z}^1(\mathcal{U}, \mathcal{O})$ such that for all $\xi \in \check{Z}^1(\mathcal{U}, \mathcal{O})$, there exist $\sigma \in S$ and $\eta \in \check{C}^0(\mathcal{W}, \mathcal{O})$ such that $\sigma = \xi + \delta\eta$ on $\mathcal{W}$.*

Proof For all $\xi \in \check{Z}^1(\mathcal{U}, \mathcal{O})$, since $\mathcal{V} \ll \mathcal{U}$, $M = \|\xi\|_{L^2(\mathcal{V})} < \infty$. By Lemma 2.6, there exist $\zeta_0 \in \check{Z}^1_{L^2}(\mathcal{U}, \mathcal{O})$, $\eta_0 \in \check{C}^0_{L^2}(\mathcal{W}, \mathcal{O})$ such that $\zeta_0 = \xi + \delta\eta_0$ on $\mathcal{W}$ and $\|\zeta_0\|_{L^2(\mathcal{U})} \leq cM$ and $\|\eta_0\|_{L^2(\mathcal{W})} \leq cM$ for some $c > 0$. Let $\varepsilon = \dfrac{1}{2}c$. By Lemma 2.2, for all $\varepsilon > 0$, there exists a closed vector subspace $A \subset \check{Z}^1_{L^2}(\mathcal{U}, \mathcal{O})$ of finite codimension such that $\|\xi\|_{L^2(\mathcal{V})} \leq \varepsilon\|\xi\|_{L^2(\mathcal{U})}$ for all $\xi \in A$. Let S be the orthogonal complement of A in $\check{Z}^1_{L^2}(\mathcal{U}, \mathcal{O})$, that is $A \oplus S = \check{Z}^1_{L^2}(\mathcal{U}, \mathcal{O})$. Suppose $\zeta_0 = \xi_0 + \sigma_0$ where $\xi_0 \in A$, $\sigma_0 \in S$ is the orthogonal decomposition. We want to construct $\zeta_i \in \check{Z}^1_{L^2}(\mathcal{U}, \mathcal{O})$, $\eta_i \in \check{C}^0_{L^2}(\mathcal{W}, \mathcal{O})$, $\xi_i \in A$, $\sigma_i \in S$ by induction such that

1. $\zeta_i = \xi_{i-1} + \delta\eta_i$ on $\mathcal{W}$
2. $\zeta_i = \xi_i + \sigma_i$
3. $\|\zeta_i\|_{L^2(\mathcal{U})} \leq 2^{-i}cM$, $\|\eta_i\|_{L^2(\mathcal{W})} \leq 2^{-i}cM$

Consider the induction step from i to $i+1$. $\|\xi_i\|_{L^2(\mathcal{V})} \leq \varepsilon_i\|\xi_i\|_{L^2(\mathcal{U})} \leq \varepsilon_i\|\zeta_i\|_{L^2(\mathcal{U})} \leq \varepsilon_i 2^{-i}cM \leq 2^{-i-1}cM$ by taking $\varepsilon_i = \dfrac{1}{2}$. By Lemma 2.6, there exist $\zeta_{i+1} \in \check{Z}^1_{L^2}(\mathcal{U},\mathcal{O})$ and $\eta_{i+1} \in \check{C}^0_{L^2}(\mathcal{W},\mathcal{O})$ such that $\zeta_{i+1} = \xi_i + \delta\eta_{i+1}$ on $\mathcal{W}$ and $\max(\|\zeta_{i+1}\|_{L^2(\mathcal{U})}, \|\eta_{i+1}\|_{L^2(\mathcal{W})}) \leq 2^{-i-1}cM$. One has orthogonal decomposition $\zeta_{i+1} = \xi_{i+1} + \sigma_{i+1}$ where $\xi_{i+1} \in A, \sigma_{i+1} \in S$. By 1 and 2, we have $\sum_{i=1}^k(\xi_i + \sigma_i) + \xi_0 + \sigma_0 = \sum_{i=1}^k(\xi_{i-1} + \delta\eta_i) + \zeta_0$ on $\mathcal{W}$. Hence $\xi_k + \sum_{i=0}^k \sigma_i = \xi + \sum_{i=0}^k \delta\eta_i$ on $\mathcal{W}$. By 2 and 3, $\|\sigma_i\|_{L^2(\mathcal{U})} \leq \|\zeta_i\|_{L^2(\mathcal{U})} - \|\xi_i\|_{L^2(\mathcal{U})} \leq \|\zeta_i\|_{L^2(\mathcal{U})} \leq 2^{-i}cM$, thus $\max(\|\xi_i\|_{L^2(\mathcal{U})}, \|\sigma_i\|_{L^2(\mathcal{U})}, \|\eta_i\|_{L^2(\mathcal{U})}) \leq 2^{-i}cM$. Hence we have $\lim_{k\to\infty}\xi_k = 0$, $\sigma = \sum_{i=0}^\infty \sigma_i \in S$ and $\eta = \sum_{i=0}^\infty \eta_i \in \check{C}^0_{L^2}(\mathcal{W},\mathcal{O})$ converge. Therefore, $\sigma = \xi + \delta\eta$ on $\mathcal{W}$. □

Lemma 2.7 shows that the natural restriction homomorphism $\check{H}^1(\mathcal{U},\mathcal{O}) \to \check{H}^1(\mathcal{W},\mathcal{O})$ has finite dimensional image.

Theorem 2.8 *Suppose $\mathcal{F}$ is a sheaf of abelian groups on a topological space X and $\mathcal{U} = (U_\alpha)_{\alpha\in A}$ is an open cover on X such that $\check{H}^1(U_\alpha,\mathcal{F}) = 0$ for every $\alpha \in A$. Then $\check{H}^1(X,\mathcal{F}) \cong \check{H}^1(\mathcal{U},\mathcal{F})$.*

Theorem 2.9 *Let X be a Riemann surface and $Y_1 \Subset Y_2$ be open sets of X. Then the restriction homomorphism $\check{H}^1(Y_2,\mathcal{O}) \to \check{H}^1(Y_1,\mathcal{O})$ has finite dimensional image.*

Proof There exists a finite family of charts $(U_i)_{1\leq i\leq n}$ on X with local coordinate z_i on U_i, and $\mathcal{W} \ll \mathcal{V} \ll \mathcal{U} \ll \mathcal{U}^*$ such that $Y_1 \subseteq \bigcup_{i=1}^n W_i = Y' \subseteq \bigcup_{i=1}^n U_i = Y'' \subseteq Y_2$ and $z_i(U_i^*), z_i(U_i), z_i(W_i)$ are discs in $\mathbb{C}$. By Lemma 2.7, the restriction map $\check{H}^1(\mathcal{U},\mathcal{O}) \to \check{H}^1(\mathcal{W},\mathcal{O})$ has finite dimensional image. By Theorem 2.5, $\check{H}^1(U_i,\mathcal{O}) = \check{H}^1(W_i,\mathcal{O}) = 0$ for all i. By Theorem 2.8, $\check{H}^1(U_i,\mathcal{O}) = \check{H}^1(W_i,\mathcal{O}) = 0$ for all i implies that $\check{H}^1(Y'',\mathcal{O}) \cong \check{H}^1(\mathcal{U},\mathcal{O})$ and $\check{H}^1(Y',\mathcal{O}) \cong \check{H}^1(\mathcal{W},\mathcal{O})$. Hence, $\check{H}^1(Y'',\mathcal{O}) \to \check{H}^1(Y',\mathcal{O})$ has finite dimensional image. The restriction map $\check{H}^1(Y_2,\mathcal{O}) \to \check{H}^1(Y_1,\mathcal{O})$ can be factored as $\check{H}^1(Y_2,\mathcal{O}) \to \check{H}^1(Y'',\mathcal{O}) \to$

$\check{H}^1(Y', \mathcal{O}) \to \check{H}^1(Y_1, \mathcal{O})$. So $\check{H}^1(Y_2, \mathcal{O}) \to \check{H}^1(Y_1, \mathcal{O})$ has finite dimensional image. □

Let X be a compact Riemann surface and $Y_1 = Y_2 = X$. Theorem 2.9 gives $\dim \check{H}^1(X, \mathcal{O}) < \infty$.

Theorem 2.10 *Let X be a Riemann surface and $Y \Subset X$. Then for all $a \in X$, there exists $f \in \mathcal{M}(Y)$ such that f has a pole at a and holomorphic on $Y - \{a\}$.*

Proof Let k be the dimension of the image of the natural restriction homomorphism $\check{H}^1(X, \mathcal{O}) \to \check{H}^1(Y, \mathcal{O})$. k is finite by Theorem 2.9. Consider the open cover $\mathcal{U} = \{U_1, U_2\}$ where U_1 is a neighborhood of a with $z(a) = 0$, $U_2 = X - \{a\}$. For $1 \leq i \leq k+1$, z^{-i} is holomorphic on $U_1 \cap U_2 = U_1 - \{a\}$. z^{-i} represent the cocycles $(\xi_i) \in \check{Z}^1(\mathcal{U}, \mathcal{O})$. Then $(\xi_i|_Y) \in \check{Z}^1(\mathcal{U} \cap Y, \mathcal{O})$ for $1 \leq i \leq k+1$ are linearly dependent modulo coboundaries. That is, there exist $c_1, \ldots, c_{k+1} \in \mathbb{C}$ not all zero such that $c_1\xi_1 + \cdots + c_{k+1}\xi_{k+1} = \delta\eta$ on $U_1 \cap U_2 \cap Y$ where $\eta = (f_1, f_2) \in \check{C}^0(\mathcal{U} \cap Y, \mathcal{O})$. So, $\sum_{i=1}^{k+1} c_i z^{-i} = f_2 - f_1$ on $U_1 \cap U_2 \cap Y$. Hence, there exists $f \in \mathcal{M}(Y)$ such that $f = f_1 + \sum_{i=1}^{k+1} c_i z^{-i}$ on $U_1 \cap Y$ and $f = f_2$ on $U_2 \cap Y = Y - \{a\}$, which agrees on $U_1 \cap U_2 \cap Y$. □

Corollary 2.11 *Let X be a compact Riemann surface and $a_1, \ldots, a_n$ be distinct points in X. Then for all $c_1, \ldots, c_n$ in $\mathbb{C}$, there exists $f \in \mathcal{M}(X)$ such that $f(a_i) = c_i$ for $1 \leq i \leq n$.*

Proof For every pair $i \neq j$, by Theorem 2.10 where $Y = X$, there exists $f_{ij} \in \mathcal{M}(X)$ with pole at a_i and holomorphic at a_j. Choose a constant $\lambda_{ij} \in \mathbb{C}^*$ such that $\lambda_{ij} \neq f_{ij}(a_j) - f_{ij}(a_k)$ for every $k = 1, \ldots, n$. Let $g_{ij} = \dfrac{f_{ij} - f_{ij}(a_j)}{f_{ij} - f_{ij}(a_j) + \lambda_{ij}} \in \mathcal{M}(X)$. Then g_{ij} is holomorphic at a_k for $k = 1, \ldots, n$, $g_{ij}(a_j) = 0$ and $g_{ij}(a_i) = 1$. Let $h_i = \prod_{i \neq j} g_{ij} \in \mathcal{M}(X)$ for $i = 1, \ldots, n$. Then $h_i(a_j) = \delta_{ij}$. Let $f = \sum_{i=1}^{n} c_i h_i \in \mathcal{M}(X)$. Then $f(a_i) = c_i$. □

2.2 Transcendence Degree of $\mathcal{M}(X)$

This section is based on the textbook [4].

Lemma 2.12 *Let D be a divisor on a Riemann surface X and $p \in X$. Then either $L(D-p) = L(D)$ or $L(D-p)$ has codimension one in $L(D)$.*

Proof Choose a local coordinate z centered at p. Let $f \in L(D)$, write $f(z) = cz^{-D(p)}+$ higher order terms, near p. Define $\alpha : L(D) \to \mathbb{C}$ by sending f to the coefficient of the term $z^{-D(p)}$. Clearly, α is linear and $\ker(\alpha) = L(D-p)$. If α is a zero map, then $L(D-p) = L(D)$. Otherwise α is onto, then $\dim L(D-p)+1 = \dim L(D)$. □

Lemma 2.13 *Let X be a compact Riemann surface. Then $\mathcal{M}(X)$ is an extension field of $\mathbb{C}$ of transcendence degree exactly one.*

Proof By Corollary 2.11, there exists a nonconstant $f \in \mathcal{M}(X)$. Since $\mathbb{C}$ is algebraically closed, the transcendence degree of $\mathcal{M}(X)$ is greater than one. Suppose the transcendence degree is at least two. Let f, g be algebraically independent elements of $\mathcal{M}(X)$. Let D be a non-negative divisor on X such that $f, g \in L(D)$. This can be obtained by choosing $D > \mathrm{div}_\infty(f)$ and $D > \mathrm{div}_\infty(g)$ where $\mathrm{div}_\infty(f) = -\sum_{\mathrm{ord}_p(f)<0} \mathrm{ord}_p(f) \cdot p$. Then $f^i g^j \in L(nD)$ for $i, j \geq 0$ and $i + j \leq n$. $L(nD)$ has every monomial of degree at most n in f and g and these monomials are linearly independent since f, g are algebraically independent. Hence, $\dim L(nD) \geq \dfrac{(n+1)(n+2)}{2} = \dfrac{n^2+3n+2}{2}$.

On the other hand, using the fact that $\dim L(D) \leq 1+\deg(D)$ for non-negative divisor D, we have $\dim L(nD) \leq 1+\deg(nD) = 1+\deg(D)n$. For n large, this gives a contradiction. It remains to show that $\dim L(D) \leq 1+\deg(D)$ for non-negative divisor D. We can prove this by induction on $\deg(D)$. If $\deg(D) = 0$, then $D = 0$, and $\dim L(D) = 1$. Suppose $\deg(D) = k \geq 1$. Choose $p \in X$ such that $D(p) \geq 1$. Then $\deg(D-p) = k-1$. By induction hypothesis,

$\dim L(D-p) \leq \deg(D-p)+1 = k$. By Lemma 2.12, $\dim L(D) \leq \dim L(D-p)+1$. Thus, $\dim L(D) \leq \dim L(D-p)+1 \leq k+1$. Therefore, $\dim L(D) \leq 1+ \deg(D)$. $\square$

By Corollary 2.10, there exist a nonconstant meromorphic function f such that it has only one pole of order n on a compact Riemann surface X. Denote $\mathbb{C}(f)$ as the field of all rational expression in the function f. Then $\mathbb{C} \subset \mathbb{C}(f) \subseteq \mathcal{M}(X)$. In fact $\mathcal{M}(X)$ is finitely generated over $\mathbb{C}$. To prove this, it suffices to show that $\mathcal{M}(X)$ is a finite algebraic extension of $\mathbb{C}(f)$. Let g be another meromorphic function on X. Let $S' = S^2 - \{\infty\} - \{\text{branch points of f}\} - \{\text{f-images of poles of g}\}$. For all $z \in S'$, there exist n distinct preimages of f, namely $w_1, \ldots, w_n$ and each $g(w_i)$ is finite. Define $\sigma_k(z) = (-1)^k \sum_{1\leq n_1 < \cdots < n_k \leq n} g(w_{n_1}) \ldots g(w_{n_k})$ for $z \in S'$.

Theorem 2.14 *Let f be a meromorphic function which has only one pole of order n on a compact Riemann surface X and g be another meromorphic function on X. then g satisfies an algebraic equation $g^n + \sigma_1(f)g^{n-1} + \cdots + \sigma_{n-1}(f)g + \sigma_n(f) = 0$ of degree n, where σ_k for $k = 1, \ldots, n$ are rational functions.*

Proof Since f is locally biholomorphic on S' and g is holomorphic on $f^{-1}(S')$, σ_k is holomorphic on S'. We want to show that the singularities of σ_k at points outside S' is at worst poles. Suppose $z_0 \notin S'$. Let m be the maximum pole order of g of $\{f^{-1}(z_0)\}$. Then $(z-z_0)^m g$ is holomorphic at each $w \in X$ with $f(w) = z_0$. Thus $(z-z_0)^{mn}\sigma_k$ is bounded, and is holomorphic at z_0. Hence, σ_k is a meromorphic function on S^2, and is a rational function.

For all w such that $f(w) \in S'$, $g^n(w) + \sigma_1(f(w))g^{n-1}(w) + \cdots + \sigma_n(f(w)) = \prod_{i=1}^n (g(w) - g(w_i)) = 0$, where $w_1, \ldots, w_n$ are the preimages of $f(w) \in S'$ via f. Let $P(f,g) = g^n + \sigma_1(f)g^{n-1} + \cdots + \sigma_{n-1}(f)g + \sigma_n(f)$. Then $P(f,g) = 0$ on $\{w \in X : f(w) \in S'\}$. Since σ_k is a rational function, $P(f,g)$ is a well-defined meromorphic function on X and $P(f,g) = 0$ on $\{w \in X : f(w) \notin S'\}$ since

$\{w \in X : f(w) \notin S'\}$ has finite numbers of points. Therefore, $P(f, g) \equiv 0$ on X. $\square$

Theorem 2.15 *Let f be a meromorphic function which has only one pole of order n on a compact Riemann surface X. Then there exists a meromorphic function g on X for which $g^n + \sigma_1(f)g^{n-1} + \cdots + \sigma_{n-1}(f)g + \sigma_n(f) = 0$ of degree n is irreducible.*

Proof Let $z_0 \in S^2$ such that $\{f^{-1}(z_0)\}$ consists of n distinct points $w_1, \ldots, w_n$ on X. By Corollary 2.11, for distinct $c_1, \ldots, c_n \in \mathbb{C}$, there exists $g \in \mathcal{M}(X)$ such that $g(w_i) = c_i$ for $1 \leq i \leq n$. It suffices to show that the polynomial $P(f, g)$ corresponding to g with rational functions as coefficients is irreducible. Suppose $P(f, g) = P_1(f, g)P_2(f, g)$. Since $P(f, g) \equiv 0$ on X, $P_1(f, g) \equiv 0$ or $P_2(f, g) \equiv 0$ on X. Suppose $P_1(f, g) \equiv 0$. Then there exists $z_1 \in S^2$ which is arbitrarily close to z_0 and is not a pole of any coefficient of $P_1(f, g)$. g has n distinct values over z_1 by continuity, and they are roots of $P_1(z_1, g)$. Hence, $\deg(P_1) = n$ and $\deg(P_2) = 0$. Therefore $P(f, g)$ is irreducible. $\square$

Theorem 2.16 *Let X be a compact Riemann surface. Then $\mathcal{M}(X)$ is finitely generated extension field of $\mathbb{C}$ of transcendence degree exactly one. Moreover, $[\mathcal{M}(X) : \mathbb{C}(f)] = n$ where f is a meromorphic function on X with only one pole of order n.*

Proof By Theorem 2.14, for any meromorphic function g, there exists an algebraic equation of degree n such that $P(f, g) = 0$. Hence $[\mathcal{M}(X) : \mathbb{C}(f)] \leq n$. By Theorem 2.15, there exists an algebraic equation of degree n such that $P(f, g) = 0$ and P is irreducible. P is an algebraic equation of minimal degree n such that $P(f, g) = 0$, thus $[\mathcal{M}(X) : \mathbb{C}(f)] = n$. Combine with Lemma 2.13, this proves that $\mathcal{M}(X)$ is finitely generated extension field of $\mathbb{C}$ of transcendence degree exactly one. $\square$

Lemma 2.17 *Let A be a divisor on a compact Riemann surface X. Let f be a nonconstant meromorphic function on X and $D = \sum_{ord_p(f)<0} -\mathrm{ord}_p(f) \cdot p$. Then there exists an integer $m > 0$ and a meromorphic function g such that $A-$ $\mathrm{div}(g) \leq mD$. Moreover, g can be taken to be a polynomial in f, that is $g = r(f)$ for some polynomial $r(t) \in \mathbb{C}[t]$*

Proof Write $A = \sum_{p\in X} A(p) \cdot p$ and $D = \sum_{ord_p(f)<0} -\mathrm{ord}_p(f) \cdot p$. Consider $p_1, \ldots, p_k$ such that $A(p_i) \geq 1$ and $\mathrm{ord}_p(f) \geq 1$. Then $(f - f(p_i))^{A(p_i)}$ has zero at p_i with order at least $A(p_i)$ and has no poles other than poles of f. Let $g = \prod_{i=1}^{k}(f - f(p_i))^{A(p_i)}$. Then $A-$ $\mathrm{div}(g) > 0$ only possible at the poles of f. Therefore, $A-$ $\mathrm{div}(g) \leq mD$ for large m. □

Corollary 2.18 *Let f and h be nonconstant meromorphic functions on a compact Riemann surface X. Then there exists a polynomial $r(t) \in \mathbb{C}[t]$ such that $r(f)h$ has no pole outside the poles of f. In this case, there exists an integer m such that $r(f)h \in L(mD)$ where $D = \sum_{ord_p(f)<0} -\mathrm{ord}_p(f) \cdot p$.* □

Proof By Lemma 2.17, let $A = -\mathrm{div}(h)$. There exists an integer $m > 0$ and a meromorphic function $g = r(f)$ such that $-\mathrm{div}(h) - \mathrm{div}(g) \leq mD$. Hence $\mathrm{div}(r(f)h) \geq -mD$, and $r(f)h \in L(mD)$. Also, $\mathrm{div}(r(f)h) \geq 0$ outside the poles of f. □

Lemma 2.19 *Fix a meromorphic function f on a compact Riemann surface. Let $D = \sum_{ord_p(f)<0} -\mathrm{ord}_p(f) \cdot p$. If $[\mathcal{M}(X) : \mathbb{C}(f)] \geq k$. Then there exists a constant m_0 such that for all $m \geq m_0$, $\dim L(mD) \geq (m - m_0 + 1)k$.*

Proof Suppose $g_1, \ldots g_k \in \mathcal{M}(X)$ which are linearly independent over $\mathbb{C}(f)$. By Corollary 2.18, there exists $r_i(t) \in \mathbb{C}[t]$ such that $h_i = r_i(f)g_i$ with no pole outside the poles of f. There exists m_0 such that $h_i \in L(m_0 D)$. Note $h_1, \ldots, h_k$ are linearly independent over $\mathbb{C}(f)$. Fix $m \geq m_0$, $\mathrm{div}(f^i h_j) = i\mathrm{div}(f) +$ $\mathrm{div}(h_j) \geq -iD - m_0 D \geq -mD$ if $0 \leq i \leq m - m_0$. Thus $f^i h_j \in L(mD)$ for $0 \leq 0 \leq$

$m - m_0, 1 \leq j \leq k$. Note that $f^i h_j$ are all linearly independent over $\mathbb{C}$. So $\dim L(mD) \geq (m - m_0 + 1)k$ for $m \geq m_0$. $\square$

2.3 The Riemann-Roch Theorem and Serre Duality

This section is based on the textbook [6].

Definition 2.20 *Let X be a compact Riemann surface. Fix $p \in X$ with z a local coordinate centered at p. Laurent tail divisor on X is the finite formal sum $\sum_{p \in X} r_p(z) \cdot p$ where $r_p(z)$ is a Laurent polynomial in coordinate z.*

Denote $\mathcal{T}(X)$ to be the set of all Laurent tail divisors on X. $\mathcal{T}(X)$ is a group under formal addition. Also denote $\mathcal{T}[D](X)$ to be the set of all Laurent tail divisors $\sum_{p \in X} r_p \cdot p$ where r_p has the form $r_p = \sum_{i=n}^{m} a_i z^i$ for $m < -D(p)$. $\mathcal{T}[D](X)$ is a subgroup of $\mathcal{T}(X)$.

Define a natural truncation map from $\mathcal{T}(X)$ to $\mathcal{T}[D](X)$, which is the truncation of each r_p by removing all terms of order $-D(p)$ and higher. If $D_1 \leq D_2$, similarly define a natural truncation map $t_{D_2}^{D_1} : \mathcal{T}[D_1](X) \to \mathcal{T}[D_2](X)$, which is the truncation of each r_p by removing all terms of order $-D_2(p)$ and higher.

Fix a meromorphic function f and a divisor D on X. Define a map μ_f^D : $\mathcal{T}[D](X) \to \mathcal{T}[D - \mathrm{div}(f)](X)$ by sending $\sum_{p \in X} r_p \cdot p$ to the truncation of $\sum_{p \in X} (f r_p) \cdot p$ by removing all terms of order $-D(p) +$ $\mathrm{ord}_p(f)$ and higher. Note that μ_f^D and $\mu_{\frac{1}{f}}^{D-div(f)}$ are inverse to each other.

Fix a divisor D on a compact Riemann surface X. Define a map α_D : $\mathcal{M}(X) \to \mathcal{T}[D](X)$ by sending f to $\sum_{p \in X} r_p \cdot p$, where r_p is the truncation of the Laurent series $f(z)$ by removing all terms of order $-D(p)$ and higher. The Mittag-Leffler problem concerns about whether a Laurent tail divisor is in the image of α_D.

Definition 2.21 *Define $H^1(D)$ to be the space of $\mathcal{T}[D](X)/im(\alpha_D)$.*

$E \in \mathcal{T}[D](X)$ is in the image of α_D if and only if the class of E in $H^1(D)$ is zero. Therefore $H^1(D)$ measures the failure of solving the Mittag-Leffler problem. We are going to prove that $H^1(D)$ is finite dimensional.

Since $L(D)$ is the space of global meromorphic functions on X with poles bounded by D, $L(D) = ker(\alpha_D)$. Thus we have an exact sequence $0 \to L(D) \to \mathcal{M}(X) \overset{\alpha_D}{\to} \mathcal{T}[D](X) \to H^1(D) \to 0$ which induces $0 \to \mathcal{M}(X)/L(D) \overset{\alpha_D}{\to} \mathcal{T}[D](X)$ $\to H^1(D) \to 0$. Also note that α_D commutes with $t^{D_1}_{D_2}$ in the sense that $\alpha_{D_2} = t^{D_1}_{D_2} \circ \alpha_{D_1}$, and α_D is compatible with μ^D_f in the sense that $\mu^D_f(\alpha_D(g)) = \alpha_{D-div(f)}$ (fg).

Lemma 2.22 *Let D_1, D_2 be divisors on a compact Riemann surface X, and $D_1 \leq D_2$. Then $\dim H^1(D_1/D_2) = [\deg(D_2) - \dim L(D_2)] - [\deg(D_1) - \dim L(D_1)]$. In particular, $H^1(D_1/D_2)$ is finite dimensional.*

Proof Consider the truncation map $t^{D_1}_{D_2} : \mathcal{T}[D_1](X) \to \mathcal{T}[D_2](X)$. Also, $L(D_1) \subseteq L(D_2)$ since $D_1 \leq D_2$. Thus we have the induced map between two exact sequence.

$$\begin{array}{ccccccc} 0 \to \mathcal{M}(X)/L(D_1) & \overset{\alpha_{D_1}}{\to} & \mathcal{T}[D_1](X) & \to & H^1(D_1) & \to 0 \\ \downarrow & & \downarrow t^{D_1}_{D_2} & & \downarrow & \\ 0 \to \mathcal{M}(X)/L(D_2) & \overset{\alpha_{D_2}}{\to} & \mathcal{T}[D_2](X) & \to & H^1(D_2) & \to 0 \end{array}$$

By the snake lemma, $0 \to ker(\mathcal{M}(X)/L(D_1) \to \mathcal{M}(X)/L(D_2)) \to ker(t^{D_1}_{D_2}) \to ker(H^1(D_1) \to H^1(D_2)) \to 0$. $ker(\mathcal{M}(X)/L(D_1) \to \mathcal{M}(X)/L(D_2)) = L(D_2)/L(D_1)$ where $\dim(L(D_2)/L(D_1)) = \dim L(D_2) - \dim L(D_1)$. $ker(t^{D_1}_{D_2})$ is the space of Laurent tail divisors $\sum_{p \in X} r_p \cdot p$ such that top term of r_p has order $< -D_1(p)$ and bottom term of r_p has order $\geq -D_2(p)$, so $\dim ker(t^{D_1}_{D_2}) = \sum_{p \in X}(D_2(p) - D_1(p)) = \deg(D_2) - \deg(D_1)$. Denote $ker(H^1(D_1) \to H^1(D_2))$ by $H^1(D_1/D_2)$. Hence we have $0 \to L(D_2)/L(D_1) \to ker(t^{D_1}_{D_2}) \to H^1(D_1/D_2) \to 0$. $H^1(D_1/D_2)$ is finite dimensional. Also, $\dim ker(t^{D_1}_{D_2}) = \dim L(D_2)/L(D_1) + \dim H^1(D_1/D_2)$ implies $\dim H^1(D_1/D_2) = [\deg(D_2) - \dim L(D_2)] - [\deg(D_1) - \dim L(D_1)]$. □

Lemma 2.23 *Let X be a compact Riemann surface. Then there is an integer M such that* $\deg(A) - \dim L(A) \leq M$ *for all divisor A on X.*

Proof By Corollary 2.10, there exists a meromorphic function f on X with only one pole of order n. Let $D = \sum_{ord_p(f)<0} -\mathrm{ord}_p(f) \cdot p$, then $\deg(D) = n$. By Proposition 2.16, $[\mathcal{M}(X) : \mathbb{C}(f)] = \deg(D)$. Then by Lemma 2.19, there exists an integer m_0 such that $\dim L(mD) \geq (m - m_0 + 1)\deg(D)$ for all $m \geq m_0$. Thus, there exists an integer $M = (m_0 - 1)\deg(D)$ such that $\deg(mD) - \dim L(mD) \leq M$ for all integers $m \geq m_0$. We can choose M large enough that $\deg(mD) - \dim L(mD) \leq M$ for all integers $m \geq 0$.

Let A be any divisor on X. By Lemma 2.17, there exists a meormorphic function g on X and an integer $m \geq 0$ such that $A - \mathrm{div}(g) \leq mD$. Let $B = A - \mathrm{div}(g)$. Then $\deg(B) = \deg(A)$ and $L(B) \cong L(A)$. Thus $\deg(A) - \dim L(A) = \deg(B) - \dim L(B)$. By Lemma 2.22, since $B \leq mD$, $\deg(B) - \dim L(B) = [\deg(mD) - \dim L(mD)] - \dim H^1(B/mD)$. Also, by the above, there exists an integer M such that $\deg(mD) - \dim L(mD) \leq M$ for all integers $m \geq 0$. Hence, $\deg(A) - \dim L(A) \leq M$. □

By Lemma 2.23, there exists a divisor A_0 such that $\deg(A_0) - \dim L(A_0)$ is maximal.

Lemma 2.24 $H^1(A_0) = 0$

Proof Suppose $H^1(A_0) \neq 0$. Then there exists $E \in \mathcal{T}[A_0](X)$ such that $E \notin im(\alpha_{A_0})$. We can increase A_0 to B such that $t^{A_0}_B(E) = 0$ for the map $t^{A_0}_B : \mathcal{T}[A_0](X) \to \mathcal{T}[B](X)$. Then $t^{A_0}_B$ induces the map $H^1(A_0) \to H^1(B)$. The nonzero class of E maps to the zero class of $t^{A_0}_B(E)$ through this map. So there exists a nonzero class of E in $H^1(A_0/B) = ker(H^1(A_0) \to H^1(B))$ and

$\dim H^1(A_0/B) \geq 1$. By Lemma 2.22, $[\deg(B) - \dim L(B)] - [\deg(A_0) - \dim(A_0)] = \dim H^1(A_0/B) \geq 1$, which contradicts to the maximality of $\deg(A_0) - \dim L(A_0)$. $\square$

Proposition 2.25 *Let D be a divisor on a compact Riemann surface X. Then $H^1(D)$ is a finite-dimensional vector space over $\mathbb{C}$.*

Proof Write $D - A_0 = P - N$ where P is a positive divisor and N is a negative divisor with disjoint support. $t_{A_0+P}^{A_0} : T[A_0](X) \to T[A_0 + P](X)$ induces the map $H^1(A_0) \to H^1(A_0 + P)$ which is onto. By Lemma 2.24, $H^1(A_0) = 0$, so $H^1(A_0 + P) = 0$. $t_{A_0+P}^{A_0+P-N} : T[A_0 + P - N](X) \to T[A_0 + P](X)$ induces the map $H^1(A_0 + P - N) \to H^1(A_0 + P)$ which is onto. By Lemma 2.22, $H^1(A_0 + P - N/A_0 + P) = ker(H^1(A_0 + P - N) \to H^1(A_0 + P))$ is finite dimensional, and $H^1(A_0+P-N/A_0+P) = H^1(A_0+P-N)$ because $H^1(A_0+P) = 0$ by above. Therefore $H^1(D) = H^1(A_0 + P - N)$ is finite dimensional. $\square$

We have proven that $H^1(D)$ is finite dimensional for a compact Riemann surface X. Therefore there exist finitely many linear conditions to be able to solve the Mittag-Leffler problem on a compact Riemann surface X.

Theorem 2.26 (The Riemann – Roch Theorem : First Form) *Let D be a divisor on a compact Riemann surface X. Then* $\dim L(D) - \dim H^1(D) = \deg(D) + 1 - \dim H^1(0)$

Proof Let D_1 and D_2 be two divisors on X such that $D_1 \leq D_2$. By Lemma2.25, $H^1(D)$ is finite dimensional, thus $\dim H^1(D_1/D_2) = ker(H^1(D_1) \to H^1(D_2)) = \dim H^1(D_1) - \dim H^1(D_2)$. Then by Lemma 2.22, $\dim H^1(D_1/D_2) = [\deg(D_2) - \dim L(D_2)] - [\deg(D_1) - \dim L(D_1)]$ implies $\dim L(D_1) - \deg(D_1) - \dim H^1(D_1) = \dim L(D_2) - \deg(D_2) - \dim H^1(D_2)$. Thus, $\dim L(D) - \deg(D) - \dim H^1(D)$ is constant for all D. In particular, when $D = 0$, we have $\dim L(D) - \deg(D) - \dim H^1(D) = 1 - H^1(0)$. Therefore $\dim L(D) - \dim H^1(D) = \deg(D) + 1 - \dim H^1(0)$ for all D. $\square$

Let D be a divisor on a compact Riemann surface X, $\omega \in L^{(1)}(-D)$ and $f \in \mathcal{M}(X)$. Write $f = \sum_{i=n}^{\infty} a_i z^i$ near p where z is a local coordinate at p. Also write $\omega = \left(\sum_{j=D(p)}^{\infty} c_j z^j\right) dz$. Then $\mathrm{Res}_p(f\omega)$ is the coefficient of $(1/z)dz$ in $f\omega = \left(\sum_{i=n}^{\infty} a_i z^i \sum_{j=D(p)}^{\infty} c_j z^j\right) dz$. This is $\sum_{j=D(p)}^{\infty} c_j a_{-1-j}$. Fix ω, this depends only on coefficients a_i of f for $i < -D(p)$, thus the residue $\mathrm{Res}_p(f\omega)$ depends only on the Laurent tail divisor $\alpha_D(f)$.

Definition 2.27 *Let D be a divisor on a compact Riemann surface X and $\omega \in L^{(1)}(-D)$. Define a residue map* $\mathrm{Res}_\omega : \mathcal{T}[D](X) \to \mathbb{C}$ *by* $\mathrm{Res}_\omega(\sum_{p \in X} r_p \cdot p) = \sum_{p\in X} \mathrm{Res}_p(r_p\omega)$

Now we have $\mathrm{Res}_\omega(\alpha_D(f)) = \sum_{p\in X}\mathrm{Res}_p(f\omega)$. By the residue theorem, $\sum_{p\in X}\mathrm{Res}_p(f\omega) = 0$. That is $\mathrm{Res}_\omega(\alpha_D(f)) = 0$. Therefore $\mathrm{Res}_\omega : \mathcal{T}[D](X) \to \mathbb{C}$ descends to $\mathrm{Res}_\omega : H^1(D) \to \mathbb{C}$. So we have Res: $L^{(1)}(-D) \to H^1(D)^*$ by sending ω to Res_ω.

Theorem 2.28 **(Serre Duality)** *Let D be a divisor on a compact Riemann surface X. The map* Res: $L^{(1)}(-D) \to H^1(D)^*$ *is an isomorphism of complex vector spaces. In particular, for any canonical divisor K on X,* $\dim H^1(D) = \dim L^{(1)}(-D) = \dim L(K-D)$.

Proof (Injectivity of Res) Suppose $\mathrm{Res}_\omega \equiv 0$ where $\omega \not\equiv 0$. Write $\omega = \sum_{i=n}^{\infty} c_i z^i dz$ where $c_n \neq 0$. Then $n = \mathrm{ord}_p(\omega) \geq D(p)$, hence $-n-1 < -D(p)$. Thus, $z^{-n-1} \cdot p \in \mathcal{T}[D](X)$, and $\mathrm{Res}_\omega(z^{-n-1}\cdot p) = \mathrm{Res}_p(z^{-n-1}\sum_{i=n}^{\infty} c_i z^i dz) = c_n \neq 0$ which leads to a contradiction. □

To prove the surjectivity of Res. We need to use the following two lemmas.

Lemma 2.29 *Let A be a divisor on a compact Riemann surface X. Let $\phi_1, \phi_2 \in H^1(A)^*$. Then there exists a positive divisor C on X and nonzero $f_1, f_2 \in L(C)$ such that* $\phi_1 \circ t_A^{A-C-\mathrm{div}(f_1)} \circ \mu_{f_1}^{A-C} = \phi_2 \circ t_A^{A-C-\mathrm{div}(f_2)} \circ \mu_{f_2}^{A-C}$ *in* $H^1(A-C)^*$.

In other words, the two maps on $\mathcal{T}[A-C](X)$

$$\begin{tikzcd}
 & \mathcal{T}[A-C-\mathrm{div}(f_1)](X) \arrow[r, "t_A^{A-C-\mathrm{div}(f_1)}"] & \mathcal{T}[A](X) \arrow[dr, "\phi_1"] & \\
\mathcal{T}[A-C](X) \arrow[ur, "\mu_{f_1}^{A-C}"] \arrow[dr, "\mu_{f_2}^{A-C}"'] & & & \mathbb{C} \\
 & \mathcal{T}[A-C-\mathrm{div}(f_2)](X) \arrow[r, "t_A^{A-C-\mathrm{div}(f_2)}"] & \mathcal{T}[A](X) \arrow[ur, "\phi_2"'] &
\end{tikzcd}$$

are equal for some C and some $f_1, f_2 \in L(C) - \{0\}$.

Proof Suppose no such C and f_i exists. Then for all divisors C, the $\mathbb{C}$-linear map $L(C) \times L(C) \to H^1(A-C)^*$ defined by sending (f_1, f_2) to $\phi_1 \circ t_A^{A-C-div f_1} \circ \mu_{f_1}^{A-C} - \phi_2 \circ t_A^{A-C-div f_2} \circ \mu_{f_2}^{A-C}$ is one-to-one. Thus, $\dim H^1(A-C) \geq 2 \dim L(C)$. By the Riemann-Roch theorem for divisor C, $\dim L(C) = \deg(C) + 1 - \dim H^1(0) + \dim H^1(C)$, thus $\dim L(C) \geq \deg(C) + 1 - \dim H^1(0)$. Therefore, for large C, $\dim H^1(A-C)$ grows at least like $2\deg(C) + a$, where a is a constant. On the other hand, by the Riemann-Roch theorem for divisor $A-C$, $\dim H^1(A-C) = \dim L(A-C) - \deg(A-C) - 1 + \dim H^1(0) \leq \dim L(A) - \deg(A) - 1 + \dim H^1(0) + \deg(C)$. Thus, for large C, $\dim H^1(A-C)$ grows at most like $\deg(C) + b$, where b is a constant. This gives a contradiction. □

Lemma 2.30 *Let D_1 be a divisor on a compact Riemann surface X and $\omega \in L^{(1)}(-D_1)$, so that $Res_\omega : \mathcal{T}[D_1](X) \to \mathbb{C}$ is well-defined. If $D_1 \leq D_2$ and Res_ω vanishes on the kernel of $t_{D_2}^{D_1} : \mathcal{T}[D_1](X) \to \mathcal{T}[D_2](X)$, then $\omega \in L^{(1)}(-D_2)$.*

Proof Suppose $\omega \notin L^{(1)}(-D_2)$. Then there exists $p \in X$ such that $n = \mathrm{ord}_p(\omega) < D_2(p)$. Consider the Laurent tail divisor $D = z^{-n-1} \cdot p$ where z is a local coordinate centered at p. $-n-1 > -D_2(p) - 1$ implies $-n-1 \geq -D_2(p)$, thus $D \in ker(t_{D_2}^{D_1})$. However, $\mathrm{Res}_\omega(D) = \mathrm{Res}_p(z^{-n-1}\omega) \neq 0$ which leads to a contradiction. □

Proof (Surjectivity of Res) Fix $\phi \in H^1(D)^*$, we want to find $\omega \in L^{(1)}(-D)$ such that $\mathrm{Res}_\omega = \phi$. Consider $\phi : H^1(D) \to \mathbb{C}$ as $\phi : \mathcal{T}[D](X) \to \mathbb{C}$ where

$\phi(\alpha_D(\mathcal{M}(X))) = 0$. Choose any meromorphic 1-form $\omega \not\equiv 0$. Let $K = \text{div}(\omega)$. Choose a divisor A such that $A \leq D$ and $A \leq K$. So $\omega \in L^{(1)}(-A)$ and $\text{Res}_\omega : \mathcal{T}[A](X) \to \mathbb{C}$ is well-defined. Now, denote $\phi_A = \phi \circ t_D^A$ on $\mathcal{T}[A](X)$. By Lemma 2.29, since $\phi_A, \text{Res}_\omega \in H^1(A)^*$, there exists a positive divisor C and nonzero $f_1, f_2 \in L(C)$ such that $\phi_A \circ t_A^{A-C-\text{div}(f_1)} \circ \mu_{f_1}^{A-C} = \text{Res}_\omega \circ t_A^{A-C-\text{div}(f_2)} \circ \mu_{f_2}^{A-C}$ on $\mathcal{T}[A-C](X)$. That is, the two maps on $\mathcal{T}[A-C](X)$

$$\begin{tikzcd}
 & \mathcal{T}[A-C-\text{div}(f_1)](X) \arrow[r, "t_A^{A-C-div(f_1)}"] & \mathcal{T}[A](X) \arrow[dr, "\phi_A"] & \\
\mathcal{T}[A-C](X) \arrow[ur, "\mu_{f_1}^{A-C}"] \arrow[dr, "\mu_{f_2}^{A-C}"'] & & & \mathbb{C} \\
 & \mathcal{T}[A-C-\text{div}(f_2)](X) \arrow[r, "t_A^{A-C-div(f_2)}"] & \mathcal{T}[A](X) \arrow[ur, "\text{Res}_\omega"'] &
\end{tikzcd}$$

are equal.

Since $\text{Res}_\omega \circ t_A^{A-C-div(f_2)} = \text{Res}_\omega$ on $\mathcal{T}[A-C-\text{div}(f_2)](X)$ and $\text{Res}_\omega \circ \mu_{f_2}^{A-C} = \text{Res}_{f_2\omega}$, we have $\phi_A \circ t_A^{A-C-\text{div}(f_1)} \circ \mu_{f_1}^{A-C} = \text{Res}_{f_2\omega}$ on $\mathcal{T}[A-C](X)$. Thus, $\phi_A \circ t_A^{A-C-div(f_1)} = \text{Res}_{f_2\omega} \circ \mu_{\frac{1}{f_1}}^{A-C-div(f_1)} = \text{Res}_{\frac{f_2}{f_1}\omega}$ on $\mathcal{T}[A-C-\text{div}(f_1)](X)$. $\frac{f_2}{f_1}\omega \in L^{(1)}(-A+C+\text{div}(f_1))$ and the above shows that $\text{Res}_{\frac{f_2}{f_1}\omega}$ vanishes on $ker(t_A^{A-C-div(f_1)})$. By Lemma 2.30, $\frac{f_2}{f_1}\omega \in L^{(1)}(-A)$. This gives $\phi_A = \text{Res}_{\frac{f_2}{f_1}\omega}$ on $\mathcal{T}[A](X)$. So $\text{Res}_{\frac{f_2}{f_1}\omega} = \phi_A = \phi \circ t_D^A$ vanishes on $ker(t_D^A)$. Since $\omega \in L^{(1)}(-A)$, by Lemma 2.30, $\frac{f_2}{f_1}\omega \in L^{(1)}(-D)$, so $\phi = \text{Res}_{\frac{f_2}{f_1}\omega}$. This completes the proof of Serre Duality. □

We can use the theorem of Serre Duality to compute the dimensions of H^1 spaces and obtain the second form of the Riemann-Roch theorem.

Theorem 2.31 (The Riemann – Roch Theorem : Second Form) *Let X be a compact Riemann surface of genus g. Then for any divisor D on X and any canonical divisor K on X, we have* $\dim L(D) - \dim L(K-D) = \deg(D) + 1 - g$.

Proof By Proposition 1.18, for any canonical divisor K on a compact Riemann surface X of genus g, $\deg(K) = 2g-2$. By Serre Duality, $\dim H^1(K) =$

$\dim L(K-K) = \dim L(0) = 1$ and $\dim H^1(0) = \dim L(K-0) = \dim L(K)$. By the Riemann-Roch theorem, $\dim L(K) - \dim H^1(K) = \deg(K) + 1 - \dim H^1(0)$. Combine with the above, $\dim H^1(0) - 1 = (2g-2) + 1 - \dim H^1(0)$. This gives $\dim H^1(0) = g$. Thus, we have $\dim L(D) - \dim L(K-D) = \deg(D) + 1 - g$. □

2.4 Holomorphic Embedding in a Projective Space

This section is based on the textbook [6].

Definition 2.32 *Let X be a Riemann surface. We say $\phi : X \to \mathbb{P}^n$ is holomorphic at $p \in X$ if there exist holomorphic functions $g_0, g_1, \ldots, g_n$ defined on X near p, and not all zero at p, such that $\phi(x) = [g_0(x) : g_1(x) : \ldots : g_n(x)]$ near p. ϕ is holomorphic if it is holomorphic for all $p \in X$.*

Fix meromorphic functions $f_0, f_1, \ldots, f_n$ on X where f_i not all identically zero. Define $\phi_f : X \to \mathbb{P}^n$ by $\phi_f(p) = [f_0(p) : f_1(p) : \ldots : f_n(p)]$ if p is not a pole of any f_i and not a zero of every f_i. Then ϕ_f is holomorphic at all such p.

Lemma 2.33 *Let X be a Riemann surface. Fix meromorphic functions $f_0, f_1, \ldots, f_n$ on X where f_i not all identically zero. Then ϕ_f defined above extends to a holomorphic map defined on all of X.*

Proof Let $k = \min_i\{\mathrm{ord}_p(f_i)\}$. It suffices to check whether ϕ_f is well-defined at p with $k \neq 0$. In a neighborhood of p, assume no f_i has a pole other than possibly at p, and no common zero to f_i other than possibly at p. Thus, by choosing a local coordinate z centered at p, $f_i(z)$ is holomorphic for $z \neq 0$ near p, and $f_i(z)$ have no common zero for $z \neq 0$ near p. For $z \neq 0$, $\phi_f(z) = [f_0(z) : \ldots : f_n(z)] = [z^{-k} f_0(z) : \ldots : z^{-k} f_n(z)] = [g_0(z) : \ldots : g_n(z)]$ which has every coordinate holomorphic near $z = 0$ and has at least one coordinate nonzero at $z = 0$. So, ϕ_f is also well-defined at $z = 0$, that is $\phi_f(0) = [g_0(0) : \ldots : g_n(0)]$. □

Proposition 2.34 *Let X be a Riemann surface. Let $\phi : X \to \mathbb{P}^n$ be holomorphic. Then there exist $(n+1)$-tuple of meromorphic functions $f = (f_0, f_1, \ldots, f_n)$ on X such that $\phi = \phi_f$. Moreover if two $(n+1)$-tuples f, g induce the same map, then there exists a meromorphic function λ on X such that $g_i = \lambda f_i$ for all i.*

Proof Let $[x_0 : \ldots : x_n]$ be homogenous coordinates of $\mathbb{P}^n$. Assume $x_0 \not\equiv 0$ on $\phi(X)$. Define $f_i = \dfrac{x_i}{x_0} \circ \phi$ on X. In particular, $f_0 = \dfrac{x_0}{x_0} \circ \phi \equiv 1$. Fix $p \in X$, in a neighborhood of p, write $\phi(z) = [g_0(z) : \ldots : g_n(z)]$ for some holomorphic functions g_i of local coordinate z centered at p. $g_0 \not\equiv 0$ by assumption. Then $f_i(z) = \dfrac{g_i(z)}{g_0(z)}$ is meromorphic at p. Clearly, $\phi = \phi_f$ where $f = (1, f_1, \ldots, f_n)$. Next, we show that ϕ_f is unique. Suppose $\phi_f = \phi_g$. Assume none of f_i, g_i is identically zero. For all $p \in X$ except finitely many zeros and poles of f_i, g_i, we have $[f_0(z) : \ldots : f_n(z)] = [g_0(z) : \ldots : g_n(z)]$ in $\mathbb{P}^n$ with none of the coordinates equals zero. Hence, there exists a nonzero $\lambda(p)$ such that $g_i(p) = f_i(p)\lambda(p)$ for all i. Since $\lambda(p) = \dfrac{g_i(p)}{f_i(p)}$, λ is holomorphic at all such p. λ is also meromorphic on all of X since it is a ratio of two global meromorphic function at all but finitely many poles. □

Definition 2.35 *Let D be a divisor on a Riemann surface X. The complete linear system of D is the set of all nonnegative divisors $E \geq 0$ which are linearly equivalent to D, that is $|D| = \{E \in \mathrm{Div}(X) : E \sim D$ and $E \geq 0\}$.*

Define $S : \mathbb{P}(L(D)) \to |D|$ by sending span of f to $\mathrm{div}(f) + D$. It is well-defined since $\mathrm{div}(cf) = \mathrm{div}(f)$ for some constant c.

Lemma 2.36 *Let X be a compact Riemann surface. Then the above map S is a 1-1 correspondence.*

Proof Let $E \in |D|$. Write $E = \mathrm{div}(f) + D$ for some meromorphic function f on X where $f \in L(D)$. There exists f such that $S(f) = \mathrm{div}(f) + D = E$. Hence S is onto.

Suppose $S(f) = S(g)$. Then $\mathrm{div}(f) + D = \mathrm{div}(g) + D$, $\mathrm{div}(\frac{f}{g}) = 0$, so $\frac{f}{g}$ has no zero and pole on X. Since X is compact, $\frac{f}{g}$ must be a nonzero constant c. Thus, f, g have the same span in $L(D)$. S is one-to-one. □

Therefore, for a compact Riemann surface, a complete linear system has a natural projective space structure.

Definition 2.37 *Let D be a divisor on a compact Riemann surface X. A linear system on X is a subset of the complete linear system $|D|$ which corresponds to a linear subspace of $\mathbb{P}(L(D))$. The dimension of a linear system is the projective dimension of the corresponding linear subspace of $\mathbb{P}(L(D))$.*

Let $\phi : X \to \mathbb{P}^n$ be a holomorphic map on a compact Riemann surface X. We can associate a linear system to ϕ as follows. Write $\phi = [f_0 : \ldots : f_n]$ where f_i are meromorphic functions on X. Let $D = -\min_i\{\mathrm{div}(f_i)\}$, so that $f_i \in L(D)$. Let $V_f = \{\sum_i a_i f_i : a_i \in \mathbb{C}\}$ which is a linear subspace of $L(D)$. Define $|\phi| = \{\mathrm{div}(g) + D : g \in V_f\}$. Then $|\phi|$ forms a linear system on X.

Lemma 2.38 *The linear system $|\phi|$ defined above is well-defined, independent of the choice of functions $f_0, f_1, \ldots, f_n$.*

Proof Suppose ϕ is also defined by $\phi = [g_0 : \ldots : g_n]$ for some meromorphic functions g_i on X. By Proposition 2.34, there exists a meromorphic function λ on X such that $g_i = \lambda f_i$ for all i. Since $\mathrm{div}(g_i) = \mathrm{div}(\lambda)+ \mathrm{div}(f_i)$, $\min_i\{\mathrm{div}(g_i)\} = \mathrm{div}(\lambda) + \min_i\{\mathrm{div}(f_i)\}$. Let $D_1 = -\min_i\{\mathrm{div}(f_i)\}$ and $D_2 = -\min_i\{\mathrm{div}(g_i)\}$. $D_1 = D_2 + \mathrm{div}(\lambda)$, thus $|D_1| = |D_2|$. Let $E \in |\phi_g|$. Write $E = \mathrm{div}(\sum_i a_i g_i) + D_2 = \mathrm{div}(\sum_i a_i \lambda f_i) + D_2 = \mathrm{div}(\sum_i a_i f_i) + \mathrm{div}(\lambda) + D_2 = \mathrm{div}(\sum_i a_i f_i) + D_1 \in |\phi_f|$. Therefore, $|\phi_f| = |\phi_g|$. □

Definition 2.39 *Let $\phi : X \to \mathbb{P}^n$ be a holomorphic map on a compact Riemann surface X. $|\phi|$ defined above is called the linear system of the map ϕ.*

Definition 2.40 *Let Q be a linear system on a compact Riemann surface X. $p \in X$ is a base point of the linear system Q if for every divisor $E \in Q$, $E \geq p$. A linear system Q is base-point-free if it has no base point.*

Lemma 2.41 *Let $\phi : X \to \mathbb{P}^n$ be a holomorphic map on a compact Riemann surface X with non-degenerate image. Then the associated linear system $|\phi|$ is base-point-free.*

Proof Write $\phi = [f_0 : \ldots : f_n]$ for some meromorphic functions f_i on X. Let $D = -\min_i\{\mathrm{div}(f_i)\}$. Fix $p \in X$. Suppose $D(p) = -\mathrm{ord}_p(f_j)$ for some j. Let $E = \mathrm{div}(f_j) + D \in |\phi|$. Then $E(p) = \mathrm{ord}_p(f_j) + D(p) = 0$. □

Proposition 2.42 *Let $Q \subset |D|$ be a base-point-free linear system of projective dimension n on a compact Riemann surface X. Then there exists a holomorphic map $\phi : X \to \mathbb{P}^n$ such that $Q = |\phi|$ and ϕ is unique up to the choice of coordinates in $\mathbb{P}^n$.*

Proof Suppose the linear system Q corresponds to a vector subspace V of $L(D)$. Then $Q = \{\mathrm{div}(f) + D : f \in V\}$. Choose basis $f_0, \ldots, f_n$ for V. Then $Q = |\phi|$ for $\phi = [f_0 : \ldots : f_n]$. It remains to show the uniqueness of ϕ. Suppose $Q = |\phi'|$ where $\phi' = [g_0 : \ldots : g_n]$. Then $|\phi'| = \{\mathrm{div}(g) + D' : g = \sum_i a_i g_i \text{ where } a_i \in \mathbb{C}\}$, where $D' = -\min_i\{\mathrm{div}(g_i)\}$. Since $|\phi| = |\phi'|$, we may change coordinates for ϕ and assume that $\mathrm{div}(f_i) + D = \mathrm{div}(g_i) + D'$ for all i. $\mathrm{div}(\frac{f_i}{g_i}) = D' - D$ is constant and independent of i. Hence, all $\frac{f_i}{g_i}$ are equal up to constant factor. By adjusting g_i by constant factors, there exists a single meromorphic function $\frac{f_i}{g_i}$. Therefore $\phi = \phi'$, up to change of coordinates in $\mathbb{P}^n$. □

Combining Lemma 2.41 and Proposition 2.42, we have a 1-1 correspondence between base-point-free linear system of projective dimension n and holomorphic map $\phi : X \to \mathbb{P}^n$ with non-degenerate image up to linear coordinate change. We denote ϕ_D as the holomorphic map associated to the complete linear system $|D|$.

Lemma 2.43 *Let D be a divisor on a compact Riemann surface X. $p \in X$ is a base point of $|D|$ if and only if $\dim L(D-p) = \dim L(D)$. Hence $|D|$ is base-point-free if and only if for all $p \in X$, $\dim L(D-p) = \dim L(D) - 1$.*

Proof Clearly, $L(D-p) \subseteq L(D)$. Let $f \in L(D)$ and $E = \operatorname{div}(f) + D \in |D|$. If p is a base point of the complete linear system $|D|$, $E(p) \geq 1$ for all $E \in |D|$. That is $\operatorname{ord}_p(f) + D(p) \geq 1$, which implies $\operatorname{div}(f) \geq -D + p$. Hence $f \in L(D-p)$, and $L(D) \subseteq L(D-p)$. Thus p is a base point of $|D|$ if and only if $L(D-p) = L(D)$. By Lemma 2.12, p is not a base point of $|D|$ if and only if $\dim L(D-p) = \dim L(D) - 1$. □

Lemma 2.44 *Let X be a compact Riemann surface. Let D be a divisor where the complete linear system $|D|$ is base-point-free. Fix distinct $p, q \in X$. Then $\phi_D(p) = \phi_D(q)$ if and only if $L(D-p-q) = L(D-p) = L(D-q)$. Hence ϕ_D is one-to-one if and only if every pair of distinct $p, q \in X$, we have $\dim L(D-p-q) = \dim L(D) - 2$.*

Proof Since $|D|$ is base-point-free, by Lemma 2.43, consider the codimension one subspace $L(D-p)$ of $L(D)$. Let $\{f_1, \dots, f_n\}$ be a basis for $L(D-p)$. Extend the basis for $L(D)$ by adding f_0 in $L(D) - L(D-p)$. Then $\operatorname{ord}_p(f_i) \geq -D(p) + 1 > -D(p)$ for all $i \geq 1$ and $\operatorname{ord}_p(f_0) = -D(p)$. Changing the basis for $L(D)$ gives a linear change of coordinates for ϕ_D. With this basis, $\phi_D = [f_0(p) : \dots : f_n(p)] = [1 : 0 : \dots : 0]$, after scaling by $z^{-D(p)}$. $\phi_D(q) = \phi_D(p)$ iff $\phi_D(q) = [1 : 0 : \dots : 0]$ iff $\operatorname{ord}_q(f_0) < \operatorname{ord}_q(f_i)$ for all $i \geq 1$. Since q is not a base point of $|D|$, this happens iff $\operatorname{ord}_q(f_0) = -D(q)$ and $\operatorname{ord}_q(f_i) > -D(q)$ for all $i \geq 1$, iff $\{f_1, \dots, f_n\}$ is a basis for $L(D-q)$ iff $L(D-p) = L(D-q)$. And so, $L(D-p) \subseteq L(D-p-q)$. Therefore this proves the first statement.

It remains to show the second statement. By Lemma 2.12, $\dim L(D-p) = \dim L(D-q) = \dim L(D) - 1$. By Lemma 2.42, either $\dim L(D-p-q) = \dim L(D-p)$ or $\dim L(D-p-q) + 1 = \dim L(D-p)$. That is, $\dim L(D-p-q)$

is either $\dim L(D) - 1$ or $\dim L(D) - 2$. By the previous argument, ϕ_D is one-to-one iff $L(D-p-q) \subsetneq L(D-p)$. This happens iff $\dim L(D-p-q) = \dim L(D) - 2$. $\square$

Lemma 2.45 *Let X be a compact Riemann surface. Let D be a divisor where the complete linear system $|D|$ is base-point-free. Assume ϕ_D is one-to-one. Fix $p \in X$. Then ϕ_D is an isomorphism onto its image near p if and only if $L(D - 2p) \neq L(D-p)$.*

Proof Same as the proof in Lemma 2.44, choose a basis $\{f_0, f_1, \ldots, f_n\}$ for $L(D)$ such that as $\mathrm{ord}_p(f_i) \geq -D(p) + 1 > -D(p)$ for all $i \geq 1$ and $\mathrm{ord}_p(f_0) = -D(p)$. With this basis, we have a coordinate z for ϕ_D. $L(D-2p) \neq L(D-p)$ iff there exists $f \notin L(D-2p)$ but $f \in L(D-p)$. That is at least one f_i has order $-D(p) + 1$, then after scaling by $z^{-D(p)}$, $f_0(z) \neq 0$, $f_i(z)(p) = 0$ for all $i \geq 1$, and at least one has a simple zero at p. By the implicit function theorem, ϕ_D is an isomorphism onto its image near p. $\square$

Proposition 2.46 *Let X be a compact Riemann surface. Let D be a divisor where the complete linear system $|D|$ is base-point-free. Then ϕ_D is a one-to-one holomorphic map and an isomorphism onto its image (that is holomorphically embedded Riemann surface in $\mathbb{P}^n$) if and only if for all $p, q \in X$, we have $\dim L(D-p-q) = \dim L(D) - 2$*

Proof By Lemma 2.12, codimension of $L(D-2p)$ in $L(D-p)$ is either 0 or 1. By Lemma 2.45, $L(D-2p) \neq L(D-p)$. Hence, $\dim L(D-2p) + 1 = \dim L(D-p)$. Since $|D|$ is base-point-free, by Lemma 2.43, $\dim L(D-p) + 1 = \dim L(D)$. Thus $\dim L(D-2p) = \dim L(D) - 2$. Combine with Lemma 2.44, for all $p, q \in X$, we have $\dim L(D-p-q) = \dim L(D) - 2$. $\square$

Proposition 2.47 *Every compact Riemann surface X can be holomorphically embedded into a projective space.*

Proof Let g be the genus of X. Fix any $p \in X$, and consider divisor $D = (2g+1) \cdot p$, then $\deg(D) = 2g+1$. First, we want to show that if $\deg(D) \geq 2g-1$, then $L(K-D) = 0$. Suppose $L(K-D) \neq 0$. We have $\deg(K-D) = \deg(K) - \deg(\mathrm{D}) \leq (2g-2) - (2g-1) = -1$. For all $f \in L(K-D)$, $\mathrm{div}(f) \geq -K+D$, so $\deg f \geq \deg(-K+D) \geq 1$ which leads to a contradiction.

Thus by Serre Duality, $H^1(D) = L(K-D) = 0$. And by the Riemann-Roch theorem, this gives $\dim L(D) = \deg(D)+1-g$. Now both $\deg(D), \deg(D-p-q) \geq 2g-1$. By the previous argument, we have $\dim L(D) = \deg(D)+1-g$ and $\dim L(D-p-q) = \deg(D-p-q)+1-g$. This implies $\dim L(D-p-q) = \dim L(D) - 2$. By Proposition 2.46, $\phi_D : X \to \mathbb{P}^n$ is a holomorphic embedding, where n is the projective dimension of the complete linear system $|D|$. □

2.5 Algebraic Curves

This section is based on the textbook [7].

Definition 2.48 *An algebraic variety $X \subset \mathbb{P}^n$ is the locus of zeros of a collection of homogenous polynomials $\{F_\alpha(x_0, \ldots, x_n)\}_{\alpha \in A}$ in $\mathbb{P}^n$. We say X is an algebraic curve if X has dimension one.*

Definition 2.49 *An analytic variety $X \subset \mathbb{P}^n$ is given locally as the zeros of a finite collection of holomorphic functions.*

Theorem 2.50 **(Chow's Theorem)** *Any analytic subvariety of projective space is algebraic.*

Theorem 2.51 *Every compact Riemann surface can be represented as a algebraic curve.*

Proof By Theorem 2.47, X can be holomorphically embedded in some $\mathbb{P}^n$. So, it is a complex submanifold in $\mathbb{P}^n$, and is a an analytic subvariety in $\mathbb{P}^n$. By the Chow's theorem, it is an algebraic curve. □

Definition 2.52 *Let $f : X \to Y$ be a continuous map between two topological spaces. f is called a covering map if for all $y \in Y$, there is an open neighbourhood V of y such that $f^{-1}(V)$ is a disjoint union $\coprod_{i \in I} U_i$ of open sets U_i such that $f|_{U_i}$ is a homeomorphism onto V for all $i \in I$.*

The cardinality of $f^{-1}(y)$ is locally constant. If Y is connected, the cardinality of $f^{-1}(y)$ is independent of $y \in Y$. f is called a finite covering if the cardinality of $f^{-1}(y)$ is finite. In particular, f is called a n-sheeted covering if the cardinality of $f^{-1}(y)$ is n. Note that any connected n-sheeted covering of $\Delta^* = \{z \in \mathbb{C} : |z| < 1\} - \{0\}$ is isomorphic to the map from Δ^* to Δ^* by sending z to z^n.

Definition 2.53 *Let $f : X \to Y$ be a continuous map between two locally compact topological spaces. Then f is called proper if for all compact sets $K \subset Y$, $f^{-1}(K)$ is compact in X.*

Note that if $f : X \to Y$ is a local homeomorphism, then f is a finite covering if and only if it is proper.

Now suppose X is a Riemann surface and $f : X \to \mathbb{C}$ is a holomorphic function and a local homeomorphism. Consider $f : X \to \mathbb{P}^1$ as local homeomorphism. Suppose $\{x_n\}$ is a sequence of points in X such that

1. $\{x_n\}$ is discrete
2. $\{f(x_n)\}$ converges to a point $a \in \mathbb{P}^1$
3. Let $D_\varepsilon = \{z \in \mathbb{C} : |z - a| < \varepsilon\}$ if $a \in \mathbb{C}$ and $D_\varepsilon = \{z \in \mathbb{C} : |z| < \dfrac{1}{\varepsilon}\} \cup \{\infty\}$ if $a = \infty$. Then for sufficiently small $\varepsilon > 0$, all but finitely many $\{x_n\}$ lie in same component of $f^{-1}(D_\varepsilon)$

Two such sequences $\{x_n\}$ and $\{y_n\}$ are equivalent if the sequence $\{z_n\}$ defined by $z_n = x_{\frac{n+1}{2}}$ for n is odd and $z_n = y_{\frac{n}{2}}$ for n is even also satisfies properties (1) to (3).

Definition 2.54 *A boundary point of X relative to f is an equivalence class of sequence $\{x_n\}$ defined above. Denote $\widetilde{X} = X \cup \{$boundary points of $X\}$.*

Let p be a boundary point of X, defined by the sequence $\{x_n\}$. Define neighbourhoods of p as follows. Let $a = \lim_{n\to\infty} f(x_n)$. For small $\varepsilon > 0$, Let $D_\varepsilon = \{z \in \mathbb{C} : |z - a| < \varepsilon\}$ if $a \in \mathbb{C}$ and $D_\varepsilon = \{z \in \mathbb{C} : |z| > \frac{1}{\varepsilon}\} \cup \{\infty\}$ if $a = \infty$. Let Ω_ε be a connected component of $f^{-1}(D_\varepsilon)$ containing all but finitely many points of $\{x_n\}$. Let $\widetilde{\Omega}_\varepsilon$ be the union of Ω_ε where Ω_ε has the property that if $\{x_n\}$ defines a boundary point p in Ω_ε then the set $\{n : x_n \notin \Omega_\varepsilon\}$ is finite. $\widetilde{\Omega}_\varepsilon$ is independent of the sequence defining the boundary point and is a neighbourhood of a boundary point. Note that this gives a Hausdorff topology. Also, f extends to a continuous map $\widetilde{f} : \widetilde{X} \to \mathbb{P}^1$ by $\widetilde{f}(p) = \lim_{n\to\infty} f(x_n)$ if p is a boundary point defined by the sequence $\{x_n\}$.

Definition 2.55 *A boundary point p of X relative to f is algebraic if the following holds. Let D_ε be a small disc around $a = \widetilde{f}(p)$ and Ω be a connected component of $f^{-1}(D_\varepsilon)$ containing all but finitely many points of a sequence defining p. Then $f(\Omega) \subset D_\varepsilon - \{a\}$ and $f : \Omega \to D_\varepsilon - \{a\}$ is a finite covering.*

Let $\Delta_R = \{z \in \mathbb{C} : |z| < R\}$ and $\Delta_R^* = \Delta_R - \{0\}$. There exists $n \geq 1$ such that $f : \Omega \to D_\varepsilon - \{a\}$ is isomorphic to $f_n : \Delta^*_{\varepsilon^{\frac{1}{n}}} \to D_\varepsilon - \{a\}$ given by $f_n(z) = a + z^n$ if $a \in \mathbb{C}$ and $f_n(z) = z^{-n}$ if $a = \infty$. Then $\widetilde{\Omega} = \Omega \cup \{p\}$ is a neighbourhood of p in $\widetilde{X}$ containing no other boundary. Since $f : \Omega \to D_\varepsilon - \{a\}$ is isomorphic to f_n defined above, there exists a homeomorphism $\phi : \Omega \to \Delta_{\varepsilon^{\frac{1}{n}}}$ with $\phi(p) = 0$ and $f \circ \phi^{-1} = f_n$ on $\Delta^*_{\varepsilon^{\frac{1}{n}}}$. ϕ is holomorphic.

Let $\widehat{X} = X \cup \{$algebraic boundary points of $X\}$. We can extend the complex structure on X to one on $\widehat{X}$ by taking (Ω, ϕ) constructed above as a chart containing an algebraic boundary point $p \in \widehat{X} - X$. Let $\widehat{f} = \widetilde{f}|_{\widehat{X}}$. Then $(\widehat{X}, \widehat{f})$ is an algebraic completion of (X, f).

Now we want to construct a Riemann surface associated to an irreducible polynomial. Let $P(x,y) = s_0(x)y^n + s_1(x)y^{n-1} + \cdots + s_n(x)$ be an irreducible polynomial with $s_i(x)$ as rational functions. Let $V = \{(x,y) \in \mathbb{C}^2 : P(x,y) = 0\}$ and $S' = \{x \in \mathbb{C} : s_0(x) = 0\} \cup \{x \in \mathbb{C} :$ there exists $y \in \mathbb{C}$ such that $P(x,y) = 0 = \dfrac{\partial P}{\partial y}(x,y)\} \cup \{\infty\} \subset \mathbb{P}^1$. Define the first projection $\pi : V \to \mathbb{C}$ by $\pi(x,y) = x$. Let $V' = V - \pi^{-1}(S')$ and $\pi' = \pi|_{V'}$. Then π' is a n-sheeted finite covering by the implicit function theorem. If D_ε is a small disc around $a \in S'$, then $\pi'|_{\pi^{-1}(D_\varepsilon - \{a\})}$ is also a finite covering. Thus $\pi^{-1}(D_\varepsilon - \{a\})$ has finitely many connected components.

Let $W_1, \ldots, W_r$ be the connected components of V' and $\pi_j = \pi|_{W_j} : W_j \to \mathbb{P}^1 - S'$. Then π_j is also a finite covering. Hence every boundary point of W_j is algebraic. Let $\widehat{\pi}_j : \widehat{W}_j \to \mathbb{P}^1$ be the algebraic completion of $\pi_j : W_j \to \mathbb{P}^1 - S'$. If $p \in \widehat{W}_j - W_j$ and $a = \widehat{\pi}_j(p)$, there exists a neighbourhood U of p and $\varepsilon > 0$ such that $\widehat{\pi}_j|_U : U \to D_\varepsilon$ is isomorphic to the map $z \mapsto a + z^m$ for some $m > 0$. Hence $\widehat{\pi}_j|_U : U \to D_\varepsilon$ is proper. For all $a \in S'$, there exists $\varepsilon > 0$ such that $\widehat{\pi}_j|_{\widehat{\pi}_j^{-1}(D_\varepsilon)} : \widehat{\pi}_j^{-1}(D_\varepsilon) \to D_\varepsilon$ is proper. So $\widehat{\pi}_j|_{W_j} : W_j \to \mathbb{P}^1 - S'$ is proper, and $\widehat{\pi}_j : \widehat{W}_j \to \mathbb{P}^1$ is proper. Therefore $\widehat{W}_j$ is compact.

Let $pr_2 : V \to \mathbb{C}$ be the second projection by $pr_2(x,y) = y$. Let $\eta = pr_2|_{V'}$ which is holomorphic on V'. Let $\eta_j = \eta|_{W_j}$ which is holomorphic on W_j.

Lemma 2.56 *Suppose $z^n + c_1 z^{n-1} + \cdots + c_n = 0$ for some $c_1, \ldots, c_n \in \mathbb{C}$ where not all $c_i = 0$. Then $|z| < 2\max_i |c_i|^{\frac{1}{i}}$.*

Proof Let $c = \max_i |c_i|^{\frac{1}{i}} > 0$. Let $w = \dfrac{z}{c}$. Then $w^n + \dfrac{c_1}{c}w^{n-1} + \cdots + \dfrac{c_n}{c^n} = 0$. Since $|c_i| \le c^i$, $|w|^n \le |w|^{n-1} + \cdots + 1$. Suppose $|w| \ge 2$. Then $1 \le \dfrac{1}{|w|} + \cdots + \dfrac{1}{|w|^n} \le \dfrac{1}{2} + \cdots + \dfrac{1}{2^n} < 1$ which gives a contradiction. Therefore, $|z| < 2c$. $\square$

Lemma 2.57 *η_j extends to a meromorphic function on $\widehat{W}_j$.*

Proof Let $a \in S'$, $p \in \widehat{W}_j$ and $\widehat{\pi}_j(p) = a$. Choose a local coordinate z at p and w at a such that $\widehat{\pi}_j$ becomes $z \mapsto z^m = w$. Let U be a neighbourhood of

p. By definitions of V and η, for $z \neq 0$, $\eta_j^n(z) + \dfrac{s_1(w)}{s_0(w)}\eta_j^{n-1}(z) + \cdots + \dfrac{s_n(w)}{s_0(w)} = 0$ where $w = \widehat{\pi}_j(z)$. Since $\dfrac{s_i(w)}{s_0(w)}$ is meromorphic at $w = 0$, there exists $c \geq 0$ and $N > 0$ such that $\left|\dfrac{s_i(w)}{s_0(w)}\right| \leq \dfrac{c}{|w|^N}$ near $w = 0$. By Lemma 2.56, $|\eta_j(z)| \leq 2\max_i \left|\dfrac{s_i(w)}{s_0(w)}\right|^{\frac{1}{i}} \leq 2\max_i \dfrac{c^{\frac{1}{i}}}{|w|^{\frac{N}{i}}} \leq \dfrac{c'}{|z|^k}$ for some constants c, c', k. Thus η_j has a meromorphic extension on $\widehat{W}_j$. $\square$

Lemma 2.58 *V' is connected.*

Proof Suppose not, then $\pi_1 : W_1 \to \mathbb{P}^1 - S'$ is a r-sheeted covering where $1 \leq r < n$. For $x \in \mathbb{P}^1 - S'$, let $\pi^{-1}(x) = \{p_1, \ldots, p_r\}$. Define $\sigma_k(x) = \sum_{1\leq i_1<\ldots<i_r\leq r} \eta_1(p_{i_1})\ldots\eta_k(p_{i_k})$. Since η_j is meromorphic on $\widehat{W}_j$, in a neighbourhood of $a \in S'$, $|\sigma_k(x)| = |\sum_{1\leq i_1<\ldots<i_r\leq r} \eta_1(p_{i_1})\ldots\eta_k(p_{i_k})| \leq c|z|^{-l} \leq c'|x-a|^{-l'}$ for some constants c, c', l, l'. Hence $\sigma_k(x)$ extends to a meromorphic function on $\mathbb{P}^1$. Let $G(x,y) = y^r + \sigma_1(x)y^{r-1} + \cdots + \sigma_r(x) = \prod_{i=1}^r (y - \eta(p_i))$. If $x \in \mathbb{P}^1 - S'$, roots of $G(x,y)$ are roots of $P(x,y)$. Hence G divides P in $\mathbb{C}(x)[y]$. Since $\deg_y(G) \geq 1$, P is not irreducible. This gives a contradiction. $\square$

Since V' is connected, $\widehat{W}$ as an algebraic completion of V' is a compact Riemann surface. Also, $\widehat{W}$ has a meromorphic function η. If $\widehat{\pi} : \widehat{W} \to \mathbb{P}^1$ is an extension of $\pi' : V' \to \mathbb{P}^1 - S'$, then $F(\widehat{\pi}(p), \eta(p)) \equiv 0$ on $\widehat{W}$.

Combining with Theorem 2.14 and Theorem 2.15, one has the following.

Theorem 2.59 *Every compact Riemann surface X can be represented as the compact Riemann surface of an irreducible polynomial $P(x,y) = 0$. More precisely, for any non-constant meromorphic function f on X, we can construct a meromorphic function g satisfying a irreducible polynomial $P(f,g) = 0$. The map $w \mapsto (f(w), g(w))$ is a conformal bijection of X onto the compact Riemann surface associated to the irreducible polynomial $P(x,y) = 0$.*

Chapter 3

Invertible Sheaves and Line Bundles

This chapter is based on the textbook [6].

3.1 Algebraic Sheaves

Definition 3.1 *Let X be a compact Riemann surface. Define a sheaf $\mathcal{O}_{X,alg}$ on X by $\mathcal{O}_{X,alg}(U) = \{f \in \mathcal{M}(X) : f$ is holomorphic for all $p \in U\}$. It is called the sheaf of regular functions on X. Also define a sheaf $\mathcal{O}_{X,alg}[D]$ on X by $\mathcal{O}_{X,alg}[D](U) = \{f \in \mathcal{M}(X)$:$\mathrm{div}(f) \geq -D$ for all $p \in U\}$ where D is a divisor on X. It is called the sheaf of rational functions with poles bounded by D on X. Finally define the sheaf of rational functions on X by $\mathcal{M}_{X,alg}(U) = \underline{\mathcal{M}_X}(U)$ for every open U. Denote this sheaf by $\mathcal{M}_{X,alg}$.*

We denote X with the classical topology by X_{an}. Recall that we have introduced some sheaves on a compact Riemann surface X with the classical topology, such as $\mathcal{O}_X$, $\mathcal{O}_X[D]$, $\mathcal{M}_X$. We denote the above by $\mathcal{O}_{X_{an}}$, $\mathcal{O}_{X_{an}}[D]$, $\mathcal{M}_{X_{an}}$ to indicate that the underlying topology is the classical topology. These are the analytic sheaves of X. Note that there are natural inclusions $\mathcal{O}_{X_{an},alg} \hookrightarrow \mathcal{O}_{X_{an}}$,

$\mathcal{O}_{X_{an},alg}[D] \hookrightarrow \mathcal{O}_{X_{an}}[D]$ and $\mathcal{M}_{X_{an},alg} \hookrightarrow \mathcal{M}_{X_{an}}$.

Definition 3.2 *Let X be a compact Riemann surface. The Zariski topology on X is the topology whose open sets are cofinite sets, that is whose complement is finite, or empty set.*

We denote X with Zariski topology by X_{Zar}. Since any Zariski open set is a classical open set, Zariski topology is a subtopology of the classical topology. Every sheaf of X_{an} induces a sheaf of X_{Zar}. In particular, we have the inclusions $\mathcal{O}_{X_{Zar},alg} \hookrightarrow \mathcal{O}_{X_{an},alg}$ and $\mathcal{O}_{X_{Zar},alg}[D] \hookrightarrow \mathcal{O}_{X_{an},alg}[D]$.

One can use Zariski topology to define Cěch cohomology groups on X_{Zar} using the same construction as before. We denote the cohomology groups of X_{Zar} by $\check{H}^n(X_{Zar}, \mathcal{F})$. On X_{Zar}, we still have Lemma 1.43. However one do not have Lemma 1.44 since X_{Zar} is not paracompact.

We now have two different cohomology on X. One is the cohomology of an algebraic sheaf with Zariski topology, another one is the cohomology of an analytic sheaf with the classical topology. Since $\mathcal{O}_{X_{an},alg}[D]$ is a subsheaf of $\mathcal{O}_{X_{an}}[D]$, this induces a map $\phi_1 : \check{H}^n(X_{an}, \mathcal{O}_{X_{an},alg}[D]) \to \check{H}^n(X_{an}, \mathcal{O}_{X_{an}}[D])$. Since $\mathcal{O}_{X_{Zar},alg}[D]$ is a subsheaf of $\mathcal{O}_{X_{an},alg}[D]$, this also induces a map ϕ_2 : $\check{H}^n(X_{Zar}, \mathcal{O}_{X_{Zar},alg}[D]) \to \check{H}^n(X_{an}, \mathcal{O}_{X_{an},alg}[D])$. Consider the map $\phi = \phi_1 \circ \phi_2$: $\check{H}^n(X_{Zar}, \mathcal{O}_{X_{Zar},alg}[D]) \to \check{H}^n(X_{an}, \mathcal{O}_{X_{an}}[D])$. In fact, ϕ provides an isomorphism between $\check{H}^n(X_{Zar}, \mathcal{O}_{X_{Zar},alg}[D])$ and $\check{H}^n(X_{an}, \mathcal{O}_{X_{an}}[D])$.

Theorem 3.3 *Let X be a compact Riemann surface. Then for any divisor D on X, the map $\phi : \check{H}^n(X_{Zar}, \mathcal{O}_{X_{Zar},alg}[D]) \to \check{H}^n(X_{an}, \mathcal{O}_{X_{an}}[D])$ are group isomorphisms for all n.*

3.2 Invertible Sheaves

From now on, we will use Zariski topology as the underlying topology. For convenience, we denote a compact Riemann surface with Zariski topology by X, instead of X_{Zar}. Also, we enote the sheaf $\mathcal{O}_{X,alg}$ of regular functions on X by $\mathcal{O}$.

Definition 3.4 *Let X be a compact Riemann surface with Zarsiski topology. A sheaf $\mathcal{F}$ on X is a sheaf of $\mathcal{O}$-module if*

1. *for every open set U of X, group $\mathcal{F}(U)$ is a $\mathcal{O}(\mathcal{U})$-module*
2. *if V is an open subset of U, then $\rho_V^U : \mathcal{F}(U) \to \mathcal{F}(V)$ is $\mathcal{O}$-linear, that is for $r \in \mathcal{O}(U)$ and $f \in \mathcal{F}(U)$, $\rho_V^U(rf) = \rho_V^U(r)\rho_V^U(f)$.*

Note that a sheaf map $\phi : \mathcal{F} \to \mathcal{G}$ between two sheaves of $\mathcal{O}$-module is a sheaf map such that $\phi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is a homomorphism between $\mathcal{O}(\mathcal{U})$-modules, for every open U. The kernel sheaf of a sheaf map of sheaves of $\mathcal{O}(\mathcal{U})$-modules is also a sheaf of $\mathcal{O}(\mathcal{U})$-module.

Let $\mathcal{F}$ be a sheaf on X and U be an open set of X. Define a sheaf $\mathcal{F}|_U$ on the space U by $\mathcal{F}|_U(V) = \mathcal{F}(V)$ for every open subset V of U.

Definition 3.5 *Let X be a compact Riemann surface and $\mathcal{F}$ be a sheaf of $\mathcal{O}$-module. We say $\mathcal{F}$ is invertible if for all $p \in X$, there exists an open neighborhood U of p such that $\mathcal{F}|_U \cong \mathcal{O}|_U$ as sheaves of $\mathcal{O}|_U$-module.*

Denote $\phi_U : \mathcal{O}|_U \to \mathcal{F}|_U$ be the above isomorphism. We call it the trivialization of $\mathcal{F}$ over $\mathcal{O}$.

Lemma 3.6 *Let X be a compact Riemann surface. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}$-module. Then $\mathcal{F}$ is invertible if and only if the following. For all $p \in X$, there exists an open neighborhood U of p and a section $f_U \in \mathcal{F}(U)$ such that for all open $V \subseteq U$, $f_V = \rho_V^U(f_U)$ generates the module $\mathcal{F}(V)$ over $\mathcal{O}(V)$, and has a trivial annihilator.*

Proof Let U be an open set in X such that $\mathcal{F}|_U \cong \mathcal{O}|_U$ as sheaves of $\mathcal{O}$-modules. We have isomorphism $\phi_V : \mathcal{O}(V) \to \mathcal{F}(V)$ for all open $V \subseteq U$, and ϕ_V commutes with the restriction map. In particular, we have $\phi_U : \mathcal{O}(U) \to \mathcal{F}(U)$ as an isomorphism. Let $f_U \in \mathcal{F}(U)$ be the image of $1 \in \mathcal{O}(U)$, then f_U is a generator for free module $\mathcal{F}(U)$ over $\mathcal{O}(U)$. Define $f_V = \rho^U_V(f_U)$ for every open $V \subseteq U$. $f_V = \rho^U_V(f_U) = \rho^U_V(\phi_U(1)) = \phi_V(\rho^U_V(1)) = \phi_V(1)$. Thus f_V generates the free module $\mathcal{F}(V)$. $\mathcal{F}(V)$ is free of rank 1 over $\mathcal{O}(U)$ using a generator f_V and it has a trivial annihilator in $\mathcal{O}(V)$, that is if $rf_V = 0$ for some $r \in \mathcal{O}(V)$, then $r = 0$.

□

f_U is called the local generator for the invertible sheaf $\mathcal{F}$ at p. The above says $\mathcal{F}$ is invertible if it has a local generator at all points of X.

Lemma 3.7 *Let X be a compact Riemann surface and D be a divisor on X. Then the sheaf $\mathcal{O}_{X,alg}[D]$ is an invertible sheaf. Moreover, the local generator for $\mathcal{O}_{X,alg}[D]$ at $p \in X$ is $z^{-D(p)}$, where z is any rational function on X with a simple zero at p.*

Proof Fix $p \in X$. Since X can be holomorphically embedded in a projective space, any global meromorphic functions on X is a rational function. Let $z \in \mathcal{M}(X)$ be a rational function with a simple zero at p. Take z as a local coordinate on X at p. Let U be a Zariski open set of X defined by removing zeros and poles of z and all points q such that $D(q) \neq 0$, except p. We want to show that $f_U = z^{-D(p)}$ is a local generator for $\mathcal{O}_{X,alg}[D]$ at p. Since $f_U \in \mathcal{O}_{X,alg}[D](U)$, we can define $f_V \in \mathcal{O}_{X,alg}[D](V)$ for every $V \subseteq U$. Each f_V has trivial annihilator. It is nonzero and multiplication is all happening in $\mathcal{M}(X)$. By Lemma 3.6, it remains to check f_V generates $\mathcal{O}_{X,alg}[D](V)$ over $\mathcal{O}(V)$ for all $V \subseteq U$. Let $g \in \mathcal{O}_{X,alg}[D](V)$. Since $V \subseteq U$, for all $q \neq p$, $\mathrm{ord}_q(g) \geq -D(q) = 0$, $\mathrm{ord}_p(g) \geq -D(p)$. $f_V = z^{-D(p)}$, thus $\mathrm{ord}_q(f_V) = 0$, $\mathrm{ord}_p(f_V) = -D(p)$. Consider $r = \dfrac{g}{f_V}$, $\mathrm{ord}_p(r) \geq 0$ for all $p \in V$. Thus, $r \in \mathcal{O}(V)$ and $g = rf_V$. This proves f_V generates $\mathcal{O}_{X,alg}[D](V)$ over $\mathcal{O}(V)$.

□

Let $\mathcal{F}$ and $\mathcal{G}$ be invertible sheaves of a compact Riemann surface X. Let $\{U_i\}$ be the collection of all open sets of X on which $\mathcal{F}$ and $\mathcal{G}$ are trivialized. $\{U_i\}$ forms an open cover of X. For any open U of X, define $\mathcal{F} \otimes_{\mathcal{O}} \mathcal{G}(U) = \{(s_i) \in \prod_i \mathcal{F}(U \cap U_i) \otimes_{\mathcal{O}(U \cap U_i)} \mathcal{G}(U \cap U_i) : s_i|_{U \cap U_i \cap U_j} = s_j|_{U \cap U_i \cap U_j} \text{ for all } i, j\}$.

Lemma 3.8 *The definition given above defines an invertible sheaf on X. We denote this sheaf as $\mathcal{F} \otimes_{\mathcal{O}} \mathcal{G}$, which is called the tensor product of the invertible sheaves $\mathcal{F}$ and $\mathcal{G}$.*

Proof Clearly, $\mathcal{F} \otimes_{\mathcal{O}} \mathcal{G}$ is a presheaf. It remains to check the sheaf axiom and the invertibility condition. Choose an open set U and an open cover $\{V_k\}$ of U. Suppose $(s_i^{(k)}) \in \mathcal{F} \otimes_{\mathcal{O}} \mathcal{G}(V_k)$ for each k which agree on $V_k \cap V_l$ for all k and l. Then $s_i^{(k)} \in \mathcal{F}(V_k \cap U_i) \otimes_{\mathcal{O}(V_k \cap U_i)} \mathcal{G}(V_k \cap U_i)$ where $\mathcal{F}, \mathcal{G}$ trivialized on U_i. Let $f_i \in \mathcal{F}(U_i)$, $g_i \in \mathcal{G}(U_i)$ be generators for $\mathcal{F}(U_i)$ and $\mathcal{G}(U_i)$ over $\mathcal{O}(U_i)$. Then $f_i \otimes g_i$ is a generator of $\mathcal{F}(U_i) \otimes_{\mathcal{O}(U_i)} \mathcal{G}(U_i)$. Write $s_i^{(k)} = r_i^{(k)} f_i \otimes g_i$ for some unique $r_i^{(k)} \in \mathcal{O}(V_k \cap U_i)$, and $r_i^{(k)}|_{V_k \cap V_l \cap U_i} = r_i^{(l)}|_{V_k \cap V_l \cap U_i}$. Since $\mathcal{O}$ satisfies the sheaf axiom, they patch together to unique $r_i \in \mathcal{O}(U \cap U_i)$ such that $r_i|_{V_k \cap U_i} = r_i^{(k)}$ for all k. Let $s_i = r_i f_i \otimes g_i \in \mathcal{F}(U \cap U_i) \otimes_{\mathcal{O}(U \cap U_i)} \mathcal{G}(U \cap U_i)$. Then (s_i) is a section of $\mathcal{F} \otimes_{\mathcal{O}} \mathcal{G}$ over U and restrict to $(s_i^{(k)})$ for all k. Uniqueness of (s_i) comes from r_i. This proves the sheaf axiom.

Consider the local generators f_i, g_i for $\mathcal{F}|_{U_i}, \mathcal{G}|_{U_i}$ respectively. Let $s_j = f_i|_{U_j \cap U_i} \otimes g_i|_{U_j \cap U_i}$ for each j. Then (s_j) is a local generator and $(s_j) \in \mathcal{F} \otimes_{\mathcal{O}} \mathcal{G}(U_i)$. This proves the invertibility condition. $\square$

Let $\mathcal{F}$ be an invertible sheaf of a compact Riemann surface X. Let $\{U_i\}$ be the collection of all open sets of X on which $\mathcal{F}$ is trivialized. $\{U_i\}$ forms an open cover of X. For any open U of X, define $\mathcal{F}^{-1}(U) = \{(s_i) \in \prod_i Hom_{\mathcal{O}(U \cap U_i)}(\mathcal{F}(U \cap U_i), \mathcal{O}(U \cap U_i)) : s_i|_{U \cap U_i \cap U_j} = s_j|_{U \cap U_i \cap U_j} \text{ for all } i, j\}$.

Lemma 3.9 *The definition given above defines an invertible sheaf on X. We denote this sheaf as $\mathcal{F}^{-1}$, which is called the inverse of the invertible sheaf $\mathcal{F}$.*

Proof Clearly, $\mathcal{F}^{-1}$ is a presheaf. It remains to check the sheaf axiom and the invertibility condition. Choose an open set U and an open cover $\{V_k\}$ of U. Suppose $(s_i^{(k)}) \in \mathcal{F}^{-1}(V_k)$ for each k which agree on $V_k \cap V_l$ for all k and l. Then $s_i^{(k)} \in Hom_{\mathcal{O}(V_k \cap U_i)}(\mathcal{F}(V_k \cap U_i), \mathcal{O}(V_k \cap U_i))$ where $\mathcal{F}$ trivialized on U_i. Let $f_i \in \mathcal{F}(U_i)$ be the generator for $\mathcal{F}(U_i)$ over $\mathcal{O}(U_i)$. Suppose $s_i^{(k)}$ is a functional sending f_i to $r_i^{(k)}$ for some unique $r_i^{(k)} \in \mathcal{O}(V_k \cap U_i)$, and $r_i^{(k)}|_{V_k \cap V_l \cap U_i} = r_i^{(l)}|_{V_k \cap V_l \cap U_i}$. Since $\mathcal{O}$ satisfies the sheaf axiom, they patch together to unique $r_i \in \mathcal{O}(U \cap U_i)$ such that $r_i|_{V_k \cap U_i} = r_i^{(k)}$ for all k. Let $s_i \in Hom_{\mathcal{O}(U \cap U_i)}(\mathcal{F}(U \cap U_i), \mathcal{O}(U \cap U_i))$ be a linear functional sending f_i to r_i. Then (s_i) is a section of $\mathcal{F}^{-1}$ over U and restrict to $(s_i^{(k)})$ for all k. Uniqueness of (s_i) comes from r_i. This proves the sheaf axiom.

Consider the local generator f_i for $\mathcal{F}|_{U_i}$. Suppose $\phi_j \in Hom_{\mathcal{O}(U_j \cap U_i)}(\mathcal{F}(U_j \cap U_i), \mathcal{F}(U_j \cap U_i))$ be the linear functional sending f_i to 1. Let $s_j \in Hom_{\mathcal{O}(U_j \cap U_i)}$ $(\mathcal{F}(U_j \cap U_i), \mathcal{F}(U_j \cap U_i))$ be a linear functional sending f_i to r_j for some $r_j \in \mathcal{O}(U_j \cap U_i)$. Then $s_j(f_i) = r_j \phi_j(f_i) = r_j$. Hence ϕ_j is a local generator and $(\phi_j) \in \mathcal{F}^{-1}(U_i)$. This proves the invertibility condition. □

Denote $Inv(X)$ as the set of isomorphism classes of invertible sheaves on X. Let $\mathcal{F}$ be an invertible sheaf. Denote its isomorphism class by $[\mathcal{F}]$. Define $[\mathcal{F}] \otimes [\mathcal{G}] = [\mathcal{F} \otimes_{\mathcal{O}} \mathcal{G}]$. To see it is well-defined, suppose $\mathcal{F}(U) \cong \mathcal{F}'(U)$ and $\mathcal{G}(U) \cong \mathcal{G}'(U)$. Then $\mathcal{F}' \otimes_{\mathcal{O}} \mathcal{G}'(U) = \{(s_i) \in \prod_i \mathcal{F}'(U \cap U_i) \otimes_{\mathcal{O}(U \cap U_i)} \mathcal{G}'(U \cap U_i) : s_i|_{U \cap U_i \cap U_j} = s_j|_{U \cap U_i \cap U_j}\} \cong \{(s_i) \in \prod_i \mathcal{F}(U \cap U_i) \otimes_{\mathcal{O}(U \cap U_i)} \mathcal{G}(U \cap U_i) : s_i|_{U \cap U_i \cap U_j} = s_j|_{U \cap U_i \cap U_j}\} = \mathcal{F} \otimes_{\mathcal{O}} \mathcal{G}(U)$.

Proposition 3.10 *Let X be a compact Riemann surface. $Inv(X)$ forms an abelian group with tensor product as group operation and $[\mathcal{O}]$ as the identity. Inverse of the class of an invertible sheaf is the class of the inverse of that invertible sheaf.*

Proof First we want to show that $\mathcal{F} \otimes_{\mathcal{O}} \mathcal{F}^{-1} \cong \mathcal{O}$ for an invertible sheaf $\mathcal{F}$. Let $\{U_i\}$ be an open cover of X such that there exists a local generator f_i of each $\mathcal{F}(U_i)$. On $U \cap U_i$, $\mathcal{F}(U \cap U_i)$ is generated by $f_i|_{U \cap U_i}$, and $\mathcal{F}^{-1}(U \cap U_i)$ is generated by the functional $\phi_i : f_i \mapsto 1$. Let $(s_i) \in \mathcal{F} \otimes_{\mathcal{O}} \mathcal{F}^{-1}(U)$. Write $s_i = r_i f_i \otimes \phi_i$ for some unique $r_i \in \mathcal{O}(U \cap U_i)$ where $r_i|_{U \cap U_i \cap U_j} = r_j|_{U \cap U_i \cap U_j}$. So, r_i patch together to give $r \in \mathcal{O}(U)$. Define a sheaf map $\varphi : \mathcal{F} \otimes_{\mathcal{O}} \mathcal{F}^{-1} \to \mathcal{O}$ by sending $(s_i) \in \mathcal{F} \otimes_{\mathcal{O}} \mathcal{F}^{-1}(U)$ to $r \in \mathcal{O}(U)$, and a sheaf map $\psi : \mathcal{O} \to \mathcal{F} \otimes_{\mathcal{O}} \mathcal{F}^{-1}$ by sending $r \in \mathcal{O}(U)$ to $(s_i) \in \mathcal{F} \otimes_{\mathcal{O}} \mathcal{F}^{-1}(U)$. Then φ and ψ are inverse to each other.

Let $\mathcal{F}$, $\mathcal{G}$, $\mathcal{H}$ be invertible sheaves. Similarly, one can check that $\mathcal{O} \otimes_{\mathcal{O}} \mathcal{F} \cong \mathcal{F}$. $\mathcal{F} \otimes_{\mathcal{O}} \mathcal{G} \cong \mathcal{G} \otimes_{\mathcal{O}} \mathcal{F}$. $(\mathcal{F} \otimes_{\mathcal{O}} \mathcal{G}) \otimes_{\mathcal{O}} \mathcal{H} \cong \mathcal{F} \otimes_{\mathcal{O}} (\mathcal{G} \otimes_{\mathcal{O}} \mathcal{H})$. Thus, $Inv(X)$ has $\mathcal{O}$ as the identity and the tensor product is commutative and associative. □

3.3 Line Bundles

Definition 3.11 *Let X be a compact Riemann surface. Let L be a set and $\pi : L \to X$ be a function. A line bundle chart for L is a bijection $\phi : \pi^{-1}(U) \to \mathbb{C} \times U$ for some open set $U \subseteq X$ such that $pr_2 \circ \phi = \pi$ on $\pi^{-1}(U)$ where pr_2 is the projection map to the second component, that is $pr_2 : \mathbb{C} \times U \to U$ by $(x, p) \mapsto p$.*

For each $p \in X$, $\pi^{-1}(p)$ is called a fiber of L. It has a vector space structure via ϕ. The projection map to the first component gives a complex coordinate z which is called the fiber coordinate of L with respect to ϕ.

Definition 3.12 *Let X be a compact Riemann surface. Let L be a set and $\pi : L \to X$ be a function. Two line bundle charts on L, $\phi_1 : \pi^{-1}(U_1) \to \mathbb{C} \times U_1$ and $\phi_2 : \pi^{-1}(U_2) \to \mathbb{C} \times U_2$ are compatible if either $U_1 \cap U_2 = \emptyset$ or the map $\phi_2 \circ \phi_1^{-1} : \mathbb{C} \times (U_1 \cap U_2) \to \mathbb{C} \times (U_1 \cap U_2)$ has the form $(x, p) \mapsto (t(p)x, p)$ for some regular nowhere zero function t on $U_1 \cap U_2$. t is called the transition function between the two line bundle charts.*

Definition 3.13 *Let X be a compact Riemann surface. Let L be a set and $\pi : L \to X$ be a function. A line bundle atlas for L is a collection of pairwise* *compatible line bundle charts for L: $\phi_i : \pi^{-1}(U_i) \to \mathbb{C} \times U_i$ where $\{U_i\}$ is an open cover of X. Two line bundle atlases for L are equivalent if every line bundle chart of one is compatible with every line bundle chart of the other. A line bundle structure for L is a maximal line bundle atlas for L. A line bundle on X is a map $\pi : L \to X$ which has a line bundle structure.*

Definition 3.14 *Let X be a compact Riemann surface. Suppose $\pi_1 : L_1 \to X$ and $\pi_2 : L_2 \to X$ are two line bundles on X. A function $\alpha : L_1 \to L_2$ is a line bundle homomorphism if*

1. $\pi_2 \circ \alpha = \pi_1$
2. *for every pair of line bundle charts $\phi_1 : \pi^{-1}(U_1) \to \mathbb{C} \times U_1$ for L_1 and $\phi_2 : \pi^{-1}(U_2) \to \mathbb{C} \times U_2$ for L_2, $\phi_2 \circ \alpha \circ \phi_1^{-1} : \mathbb{C} \times (U_1 \cap U_2) \to \mathbb{C} \times (U_1 \cap U_2)$ has the form $(x, p) \mapsto (f(p)x, p)$ for some regular nowhere zero function f on $U_1 \cap U_2$.*

Definition 3.15 *A line bundle isomorphism is a line bundle homomorphism which has an inverse. Denote $LB(X)$ as the set of isomorphism classes of line bundles on X.*

Proposition 3.16 *Let X be a compact Riemann surface and $\{U_i\}$ be an open cover of X. Given nowhere zero regular functions $\{t_{ij}\}$ on $U_i \cap U_j$ for every pair of (i, j), satisfying the cocycle condition. Then there exists a line bundle L, unique up to isomorphism, with line bundle charts $\phi_i : \pi^{-1}(U_i) \to \mathbb{C} \times U_i$ having t_{ij} as transition functions. In terms of fiber coordinates z_i, we have $z_i = t_{ij} z_j$ on $U_i \cap U_j$ for every pair of (i, j).*

Proof We want to construct a line bundle using the transition functions. Let $\widetilde{L} = \coprod_i (\mathbb{C} \times U_i)$. Define partition of $\widetilde{L}$ by identifying $(s, p) \in \mathbb{C} \times U_j$ and $(t_{ij}s, p) \in \mathbb{C} \times U_i$ to be the same partition for $p \in U_i \cap U_j$. Let L be the set of partition subsets. Then there is a natural map sending (s,p) to its partition subset. Now, we want to make L to be a line bundle over X. We require that $(s,p) \in \mathbb{C} \times U_i$ identified exactly with $(t_{ii}s,p) \in \mathbb{C} \times U_i$. That is $t_{ii} \equiv 1$ for all U_i. We also need $t_{ji}t_{ij} \equiv 1$ on $U_i \cap U_j$, so that $(s,p) \in \mathbb{C} \times U_i$ identified with $(t_{ij}s,p) \in \mathbb{C} \times U_j$ and then identified back with $(t_{ji}t_{ij}s,p) = (s,p) \in \mathbb{C} \times U_i$. With one more condition: $t_{ki}t_{ij}t_{jk} \equiv 1$ on $U_i \cap U_j \cap U_k$, one can check that $t_{i_0i_1}t_{i_1i_2}\ldots t_{i_{n-1}i_n} \equiv 1$ for any $n > 3$. These three conditions are the cocycle condition of t_{ij}. It ensures that $(s,p) \in \mathbb{C} \times U_i$ is identified exactly with $(t_{ij}s,p) \in \mathbb{C} \times U_j$. The composition map $\mathbb{C} \times U_i \hookrightarrow \widetilde{L} \to L$ is injective. Let L_i be the image of $\mathbb{C} \times U_i$ in L, then the above composition is a bijection between $\mathbb{C} \times U_i$ and L_i. Denote $\phi_i : L_i \to \mathbb{C} \times U_i$ to be the inverse. Then ϕ_i is a line bundle chart on L.

It remains to check that these charts are compatible to each other. Suppose ϕ_i and ϕ_j are two line bundle charts. Let $(s,p) \in \mathbb{C} \times U_j$. Then $\phi_i(\phi_j^{-1}(s,p)) = (t_{ij}s,p) \in \mathbb{C} \times U_i$. Since t_{ij} is a nowhere zero regular function on $U_i \cap U_j$. ϕ_i, ϕ_j are compatible. □

Proposition 3.16 says that we can construct a line bundle by a collection of transition functions satisfying cocycle condition. The following examples show how to construct tangent line bundle and canonical line bundle by using transition functions.

Example Let X be a compact Riemann surface, and $\{\phi_i : U_i \to V_i\}$ be an atlas on X. Let $T_{ij} = \phi_i \circ \phi_j^{-1}$. Then T_{ij} is a biholomorphic function and T'_{ij} is nowhere zero. Let $t_{ij} = T'_{ij} \circ \phi_i|_{U_i \cap U_j}$ which is a nowhere zero holomorphic function on $U_i \cap U_j$. Also, $\{t_{ij}\}$ satisfy cocycle condition. Thus, this defines a line bundle, called the tangent line bundle T_X on X.

Example Let X be a compact Riemann surface. $\{\phi_i : U_i \to V_i\}$ be an atlas on X. Let $T_{ij} = \phi_i \circ \phi_j^{-1}$. Then T_{ij} is a biholomorphic function and T'_{ij} is nowhere zero. $T_{ij} \circ T_{ji} = id$, so $T'_{ji} = \dfrac{1}{T'_{ij}}$. Let $t_{ij} = \dfrac{1}{T'_{ij}} \circ \phi_i|_{U_i \cap U_j} = T'_{ji} \circ \phi_i|_{U_i \cap U_j}$ which is a nowhere zero holomorphic function on $U_i \cap U_j$. Also, $\{t_{ij}\}$ satisfy cocycle condition. Thus, this defines a line bundle, called the canonical bundle K_X on X.

Definition 3.17 *Let X be a compact Riemann surface. Let $\pi : L \to X$ be a line bundle on X and U be an open set of X. A regular section of L over U is a function $s : U \to L$ such that*

1. *for every $p \in U$, $s(p)$ lies in the fiber of L over p, that is, $\pi \circ s = id_U$*
2. *for every line bundle chart $\phi : \pi^{-1}(V) \to \mathbb{C} \times V$ for L, the composition $pr_1 \circ \phi \circ s|_{U\cap V} : U \cap V \to \mathbb{C}$ is a regular function on $U \cap V$.*

Denote $\mathcal{O}_{X,alg}\{L\}(U)$ as the set of regular sections of L over U.

Proposition 3.18 *Let X be a compact Riemann surface. Let $\pi : L \to X$ be a line bundle on X. Then $\mathcal{O}_{X,alg}\{L\}$ is an invertible sheaf on X.*

Proof Clearly, $\mathcal{O}_{X,alg}\{L\}$ is a presheaf. It remains to prove the sheaf axiom and the invertibility condition. Fix an open set U and an open cover $\{V_i\}$ of U. Let s_i be a regular section of L over V_i for each i and $s_i = s_j$ on $V_i \cap V_j$. Define $s : U \to L$ by $s(p) = s_i(p)$ if $p \in V_i$. It is well-defined and satisfies $\pi \circ s = id_U$. It remains to check that $pr_1 \circ \phi \circ s|_{U\cap W} : U \cap W \to \mathbb{C}$ is a regular function for all line bundle chart ϕ. Let $f_i = pr_1 \circ \phi \circ s_i|_{V_i \cap W} : V_i \cap W \to \mathbb{C}$. Then $f_i \in \mathcal{O}(V_i \cap W)$. Since $s_i = s_j$ on $V_i \cap V_j$, $f_i = f_j$ on $V_i \cap V_j \cap W$. Since $\mathcal{O}$ is a sheaf, f_i patch together to $f \in \mathcal{O}(U \cap W)$. By construction, $f = pr_1 \circ \phi \circ s|_{U\cap W} \in \mathcal{O}(U \cap W)$. Therefore s is a regular section of L over U. This proves the sheaf axiom.

Fix a line bundle chart $\phi : \pi^{-1}(U) \to \mathbb{C} \times U$ for L and an open set U. For each open subset $V \subseteq U$, define $\alpha : \mathcal{O}(V) \to \mathcal{O}_{X,alg}\{L\}(V)$ by sending f to s_f, where s_f is a regular section from V to L by sending $p \in V$ to $\phi^{-1}(f(p), p) \in L$. And define $\beta : \mathcal{O}_{X,alg}\{L\}(V) \to \mathcal{O}(V)$ by sending s to f_s, where $f_s = pr_1 \circ \phi \circ s$. Then $\alpha \circ \beta(s)(p) = \phi^{-1}(pr_1 \circ \phi \circ s(p), p) = \phi^{-1}(\phi(s(p)), p) = s(p)$. $\beta \circ \alpha(f)(p) = pr_1 \circ \phi \circ$

$\phi^{-1}(f(p), p) = f(p)$. Hence α, β are inverse to each other, thus $\mathcal{O}_{X,alg}\{L\}(V) \cong \mathcal{O}(V)$ as $\mathcal{O}$-modules. Since they are compatible with restriction maps, we also have $\mathcal{O}_{X,alg}\{L\}|_U \cong \mathcal{O}|_U$ as a sheaf isomorphism. This proves the invertibility condition. □

The above proposition induces a map $\mathcal{O}\{-\} : LB(X) \to Inv(X)$ by sending L to $\mathcal{O}_{X,alg}\{L\}$.

Definition 3.19 *Let X be a compact Riemann surface. Let $\pi : L \to X$ be a line bundle on X. A rational section of L is a regular section $s : U \to L$ on a Zariski open set U of X. In other words, there is a finite set $A \subset X$ and a function $s : X - A \to L$ such that*

1. *for every $p \in X - A$, $s(p)$ lies in the fiber of L over p, that is, $\pi \circ s = id_{X-A}$*
2. *for every line bundle chart $\phi : \pi^{-1}(V) \to \mathbb{C} \times V$ for L, the composition $pr_1 \circ \phi \circ s|_V : V \to \mathbb{C}$ is a rational function on V.*

Definition 3.20 *Let X be a compact Riemann surface. Let $\pi : L \to X$ be a line bundle on X. Let s be a rational section of L. The order of s at a point $p \in X$, denoted by $\mathrm{ord}_p(s)$, is the order of rational function $f = pr_1 \circ \phi \circ s$ where $\phi : \pi^{-1}(U) \to \mathbb{C} \times U$ is any line bundle chart of L with U containing p.*

We have to check that the above definition is independent of the choice of line bundle chart. Let $\phi' : \pi^{-1}(V) \to \mathbb{C} \times V$ be another line bundle chart with a nowhere zero regular transition function t between the two line bundle charts. Then $f' = pr_1 \circ \phi' \circ s$ is exactly tf. $\mathrm{ord}_p(f') = \mathrm{ord}_p(tf) = \mathrm{ord}_p(t) + \mathrm{ord}_p(f) = \mathrm{ord}_p(f)$.

Definition 3.21 *The divisor of a rational section s is $\mathrm{div}(s) = \sum_{p \in X} \mathrm{ord}_p(s) \cdot p$.*

Proposition 3.22 *Let L be a line bundle on a compact Riemann surface X and s_1, s_2 be two rational sections of L. Then $\mathrm{div}(s_1) \sim \mathrm{div}(s_2)$.*

Proof Fix a line bundle chart $\phi : \pi^{-1}(U) \to \mathbb{C} \times U$ for L. Let $f_i = pr_1 \circ \phi \circ s_i$ for $i = 1, 2$ and $g = \dfrac{f_1}{f_2}$. For any line bundle chart $\phi' : \pi^{-1}(U') \to \mathbb{C} \times U'$, denote the transition function from ϕ to ϕ' by t. Then $f_i' = pr_1 \circ \phi' \circ s_i$ is tf_i and we also have $g = \dfrac{f_1'}{f_2'}$. Hence g is a rational function and is independent of line bundle chart. For all $p \in U \cap U'$, $\mathrm{ord}_p(s_1) = \mathrm{ord}_p(f_1) = \mathrm{ord}_p(g) +\ \mathrm{ord}_p(f_2) = \mathrm{ord}_p(g) +\ \mathrm{ord}_p(s_2)$. Since g is independent of line bundle chart, the above holds for all $p \in X$. Thus, $\mathrm{div}(s_1) \sim \mathrm{div}(s_2)$. $\square$

Recall that the Picard group $\mathrm{Pic}(X) = \mathrm{Div}(X)/\mathrm{PDiv}(X)$. By the above proposition, we obtain a map [div]: $LB(X) \to \mathrm{Pic}(X)$ by sending L to $[\mathrm{div}(s)]$.

3.4 Isomorphic Representations of the Picard Group

Let X be a compact Riemann surface with Zariski topology. Using Zariski topology, we have the 1^{st} cohomology group $\check{H}^1(X, \mathcal{O}^*_{X,alg})$ where $\mathcal{O}^*_{X,alg}$ is the algebraic sheaf of nowhere zero regular functions, that is $\mathcal{O}^*_{X,alg}(U) = \{f \in \mathcal{O}_{X,alg}(U) : f$ has no zeros on $U\}$. By Proposition 3.10, we have the group of isomorphism classes of invertible sheaves $Inv(X)$. We also have the set of isomorphism classes of line bundles $LB(X)$. Recall that the Picard group is the group of divisors on X modulo the subgroup of principal divisors. We are going to prove the isomorphisms between the following groups: $\check{H}^1(X, \mathcal{O}^*_{X,alg})$, $Inv(X)$, $LB(X)$, $\mathrm{Pic}(X)$.

Recall that the analytic sheaf $\mathcal{D}\mathrm{iv}_X$ is defined by $\mathcal{D}\mathrm{iv}_X(U) = \{$divisors with discrete support in $U\}$. We now define an algebraic sheaf $\mathcal{D}\mathrm{iv}_{X,alg}$ by $\mathcal{D}\mathrm{iv}_{X,alg}(U) = \{$divisors with finite support in $U\}$. Let $\mathcal{M}^*_{X,alg}$ be a constant algebraic sheaf defined by $\mathcal{M}^*_{X,alg}(U) = \underline{\mathcal{M}^*_X}(U)$. Define a sheaf map div: $\mathcal{M}^*_{X,alg} \to \mathcal{D}\mathrm{iv}_{X,alg}$ by sending a nowhere zero rational function on U to the part of its divisor supported on U. Then $\mathcal{O}^*_{X,alg}$ is the kernel sheaf of the sheaf map div.

Lemma 3.23 *Let X be a compact Riemann surface with Zariski topology. The map* div: $\mathcal{M}^*_{X,alg} \to \mathcal{D}iv_{X,alg}$ *is an onto map of sheaves.*

Proof Fix a Zariski open set U containing $p \in X$ and $D \in \mathcal{D}iv_{X,alg}(U)$. If $D(p) = 0$, Define V by deleting support of D from U. Then V is a Zariski open set because D has finite support in U, also $D = 0 = \text{div}(1)$ when restricted to V. If $D(p) \neq 0$, since X can be holomorphically embedded in a projective space, any global meromorphic functions on X is a rational function, and there is a rational function f on X such that $\text{ord}_p(f) = 1$. Define V by deleting support of D and zeros and poles of f from U, except p. Then V is a Zariski open set, and $D = D(p) \cdot p = \text{div}(f^{D(p)})$ when restricted to V. $\square$

Proposition 3.24 *Let $\underline{G}$ be a constant sheaf on a Riemann surface X. Then for $n \geq 1$, $\check{H}^n(X_{Zar}, \underline{G}) = 0$.*

Proposition 3.25 *Let X be a compact Riemann surface. There is a group isomorphism between* $\text{Pic}(X)$ *and* $\check{H}^1(X, \mathcal{O}^*_{X,alg})$.

Proof The kernel sheaf of div: $\mathcal{M}^*_{X,alg} \to \mathcal{D}iv_{X,alg}$ is $\mathcal{O}^*_{X,alg}$, and by Lemma 3.23, div is onto. Thus we have the short exact sequence of sheaves $0 \to \mathcal{O}^*_{X,alg} \to \mathcal{M}^*_{X,alg} \overset{div}{\to} \mathcal{D}\text{iv}_{X,alg} \to 0$. This induces a long exact sequence of cohomology $0 \to \mathcal{O}^*_{X,alg}(X) \to \mathcal{M}^*_{X,alg}(X) \overset{div}{\to} \mathcal{D}\text{iv}_{X,alg}(X) \overset{\Delta}{\to} \check{H}^1(X, \mathcal{O}^*_{X,alg}) \to 0$ since $\check{H}^1(X, \mathcal{M}^*_{X,alg}) = 0$ by Proposition 3.24. Also, $\mathcal{O}^*_{X,alg}(X) \cong \mathbb{C}^*$, $\mathcal{M}^*_{X,alg} \cong \mathcal{M}(X) - \{0\}$ and $\mathcal{D}\text{iv}_{X,alg}(X) \cong \text{Div}(X)$. The exact sequence becomes $0 \to \mathbb{C}^* \to \mathcal{M}(X) - \{0\} \overset{div}{\to} \text{Div}(X) \overset{\Delta}{\to} \check{H}^1(X, \mathcal{O}^*_{X,alg}) \to 0$. So, $\check{H}^1(X, \mathcal{O}^*_{X,alg}) \cong \text{Div}(X)/im(\text{div}) \cong \text{Div}(X)/\text{PDiv}(X) = \text{Pic}(X)$. $\square$

The above isomorphism is induced by the connecting homomorphism. We denote this isomorphism by $\Delta : \text{Pic}(X) \to \check{H}^1(X, \mathcal{O}^*_{X,alg})$.

By Lemma 3.7, $\mathcal{O}_{X,alg}[D]$ is an invertible sheaf. For simplicity, denote $\mathcal{O}_{X,alg}[D]$ by $\mathcal{O}[D]$. One can check that if $D_1 \sim D_2$, then $\mathcal{O}_{X,alg}[D_1] \cong \mathcal{O}_{X,alg}[D_2]$ where the isomorphism is given by multiplication of f where $\text{div}(f) = D_1 - D_2$. Therefore, we have a well-defined map $\mathcal{O}[-] : \text{Pic}(X) \to Inv(X)$.

Proposition 3.26 *Let X be a compact Riemann surface. Then the map $\mathcal{O}[-]$: $Pic(X) \to Inv(X)$ is an isomorphism of groups.*

Proof First we need to check that $\mathcal{O}[-]$ is a homomorphism of groups, that means to show $\mathcal{O}[D_1 + D_2] \cong \mathcal{O}[D_1] \otimes_{\mathcal{O}} \mathcal{O}[D_2]$. By Lemma 3.7, $\mathcal{O}$ is locally generated by rational function f with $\text{div}(f) = -D$. Choose an open cover $\{U_i\}$ of X such that both $\mathcal{O}[D_1]$ and $\mathcal{O}[D_2]$ are trivialized on U_i. Let $f_i^{(1)}$ and $f_i^{(2)}$ be local generator of $\mathcal{O}[D_1]$ and $\mathcal{O}[D_2]$ on U_i. Thus, $\text{div}(f_i^{(1)} f_i^{(2)}) = -D_1 - D_2$ on U_i , and $f_i^{(1)} f_i^{(2)}$ is a local generator for $\mathcal{O}[D_1 + D_2]$ on U_i. Define a bilinear map from $\mathcal{O}[D_1] \times \mathcal{O}[D_2]$ to $\mathcal{O}[D_1 + D_2]$ by multiplication and extends bilinearly. It descends to the sheaf map from $\mathcal{O}[D_1] \otimes_{\mathcal{O}} \mathcal{O}[D_2]$ to $\mathcal{O}[D_1 + D_2]$ which sends the local generator $f_i^{(1)} \otimes f_i^{(2)}$ for $\mathcal{O}[D_1] \otimes_{\mathcal{O}} \mathcal{O}[D_2]$ to the local generator $f_i^{(1)} f_i^{(2)}$ for $\mathcal{O}[D_1 + D_2]$. Hence this map is an isomorphism. This proves that $\mathcal{O}[-]$ is a group homomorphism.

It remains to check that $\mathcal{O}[-]$ is one-to-one and onto. Since $\mathcal{O}$ is the identity in the group $Inv(X)$, it suffices to show that if $\mathcal{O}[D] \cong \mathcal{O}$ as sheaves of $\mathcal{O}$-modules, then $[D] = 0$ in $\text{Pic}(X)$. Since $\mathcal{O}$ has a global generator 1, $\mathcal{O}[D]$ also has a global generator f. $f|_U$ is a local generator for every $\mathcal{O}[D](U)$ over $\mathcal{O}(U)$. That is $\mathcal{O}[D](U)$ consists of multiples of $f|_U$ by elements in $\mathcal{O}(U)$. Hence $\mathcal{O}[D] = \mathcal{O}[-\text{div}(f)]$. This implies $D = -\text{div}(f)$. Therefore $[D] = 0$ in $Pic(X)$. This proves that $\mathcal{O}[-]$ is one-to-one.

Suppose $\mathcal{F} \in Inv(X)$. Let $\{U_i\}$ be an open cover such that $\mathcal{F}$ trivialized. Let f_i be a local generator for $\mathcal{F}|_{U_i}$. Then there exists $t_{ij} \in \mathcal{O}^*(U_i \cap U_j)$ satisfying cocycle condition such that $f_i = t_{ij} f_j$ on $U_i \cap U_j$. Fix index 0, consider $t_{i0} \in \mathcal{O}^*(U_0 \cap U_i)$ for each i. Define a divisor D on X by $D(p) = -\text{ord}_p(t_{i0})$ if $p \in U_i$. It is well-defined because if $p \in U_i \cap U_j$, $t_{i0} = t_{ij} t_{j0}$, then $\text{ord}_p(t_{i0}) = \text{ord}_p(t_{j0})$

since $t_{ij} \in \mathcal{O}^*(U_i \cap U_j)$. Now, it remains to check $\mathcal{O}[D] \cong \mathcal{F}$. On each U_i, define a map: $\mathcal{O}[D](U_i) \to \mathcal{F}(U_i)$ by sending the local generator t_{i0} for $\mathcal{O}[D](U_i)$ to the local generator f_i for $\mathcal{F}(U_i)$, and extend $\mathcal{O}(U_i)$-linearly. This is compatible with the restriction map because t_{ij} satisfy cocycle condition.

Since $t_{i0} = t_{ij}t_{j0}$ and $f_i = t_{ij}f_j$ on $U_i \cap U_j$, if $U \subset U_i \cap U_j$, the map on U_i by sending t_{i0} to f_i and the map on U_j by sending t_{j0} to f_j induce the same map on U. By the sheaf axiom, it gives an isomorphism from $\mathcal{O}[D](U)$ to $\mathcal{F}(U)$ for every open U. This proves that $\mathcal{O}[-]$ is onto. □

Define a map $H_L : LB(X) \to \check{H}^1(X, \mathcal{O}^*_{X,alg})$ by sending a line bundle L to the cohomology class of 1-cocycle (t_{ij}) where $\{t_{ij}\}$ are the transition functions of a line bundle atlas of X. One need to check that H_L independent of the choice of line bundle atlas.

Lemma 3.27 *For a line bundle L on a compact Riemann surface X, the cohomology class $H_L(L) \in \check{H}^1(X, \mathcal{O}^*_{X,alg})$ is well-defined, independent of line bundle atlas used to define L.*

Proof Fix an open cover $\mathcal{U} = \{U_i\}$ of X. Let $\phi_i : \pi^{-1}(U_i) \to \mathbb{C} \times U_i$ and $\phi'_i : \pi^{-1}(U_i) \to \mathbb{C} \times U_i$ be two line bundle charts. $\phi'_i \circ \phi_i^{-1}(x,p) = (r_i x, p)$ where $r_i \in \mathcal{O}^*(U_i)$. Then 1-cocycle defined by ϕ'_i differs from ϕ_i by the coboundary $(\dfrac{r_i}{r_j})$. So, we have the same class in $\check{H}^1(\mathcal{U}, \mathcal{O}^*_{X,alg})$, so as in $\check{H}^1(X_{Zar}, \mathcal{O}^*_{X,alg})$. This proves that H_L is independent of the choice of the line bundle atlas.

Suppose there are two open covers. Since any two open covers have a common refinement. It suffices to check the independence under refinement. Let $\mathcal{V} = \{V_k\}$ be a refinement of $\mathcal{U} = \{U_i\}$ with the refining map r, that is $V_k \subset U_{r(k)}$. Use the same line bundle charts restricted to $\mathcal{V}$. Then 1-cocycle $H_L(L)$ defined by the open cover $\mathcal{U}$ maps to 1-cocycle defined by the open cover $\mathcal{V}$ through the map $H(r)$. So, they are equal in the limit group $\check{H}^1(X_{Zar}, \mathcal{O}^*_{X,alg})$. This proves that H_L is independent of the choice of the open cover. □

Lemma 3.28 *The mapping $H_L : LB(X) \to \check{H}^1(X, \mathcal{O}^*_{X,alg})$ is a bijection*

Proof By Proposition 3.16, given $t_{ij} \in \mathcal{O}^*(U_i \cap U_j)$ satisfying cocycle condiiton, there exists a unique line bundle up to isomorphism. Since (t_{ij}) is a 1-cocycle for sheaf $\mathcal{O}^*_{X,alg}$, it satisfies cocycle condition. This proves that H_L is onto.

Suppose two line bundles $\pi_1 : L_1 \to X$ with line bundle charts $\phi_i : \pi^{-1}(V_j) \to \mathbb{C} \times V_j$ and $\pi_2 : L_2 \to X$ with line bundle charts $\psi_i : \pi^{-1}(W_k) \to \mathbb{C} \times W_k$ map to the same cohomology class under H_L. Let $\{U_i\}$ be a common refinement of the two open covers of line bundle atlas of L_1 and L_2. Then the transition functions (t_{ij}) and (s_{ij}) induced from transition functions of L_1 and L_2 respectively give the same cohomology class in $\check{H}^1(X_{Zar}, \mathcal{O}^*_{X,alg})$. That is $t_{ij}\dfrac{r_i}{r_j} = s_{ij}$ for some $r_i \in \mathcal{O}^*(U_i)$. Now, we define a line bundle automorphism $r_i : \mathbb{C} \times U_i \to \mathbb{C} \times U_i$ by $r_i(x,p) = (r_i x, p)$. Consider $\xi_i = r_i \circ \phi_i$. Then $\xi_i : \pi^{-1}(U_i) \to \mathbb{C} \times U_i$ is a line bundle chart. Since $\xi_i \circ \phi_i^{-1} = r_i$, ξ_i is compatible with ϕ_i. Let (u_{ij}) be transition functions of charts ξ_i. Then $\xi_i \circ \xi_j^{-1} = r_i \circ \phi_i \circ \phi_j^{-1} \circ r_j^{-1}$, thus $u_{ij} = t_{ij}\dfrac{r_i}{r_j} = s_{ij}$. Therefore, there are line bundle atlases for both L_1, L_2 with the same open cover and the same transition functions, so $L_1 \cong L_2$. This proves that H_L is one-to-one. □

Using the bijection $H_L : LB(X) \to \check{H}^1(X, \mathcal{O}^*_{X,alg})$, we can put an abelian group structure on $LB(X)$ by defining group operation via H_L. Then we have H_L as a group isomorphism.

Define a map $H_I : Inv(X) \to \check{H}^1(X, \mathcal{O}^*_{X,alg})$ as follows. Suppose $\mathcal{F} \in Inv(X)$. Let $\mathcal{U} = \{U_i\}$ be an open cover of X such that $\mathcal{F}$ is trivialized on each U_i. Let f_i be the generators on U_i. Write $f_i = t_{ij} f_j$ on each U_{ij}. Then $t_{ij} \in \mathcal{O}^*(U_i \cap U_j)$ and (t_{ij}) is a 1-cocycle for the sheaf $\mathcal{O}^*$. Hence it represents a class in $\check{H}^1(X, \mathcal{O}^*_{X,alg})$. Define $H_I(\mathcal{F})$ to be this class. Note that it is independent of the choice of local generators and open covers.

In fact, one can show that $H_I \circ \mathcal{O}[-] = \Delta$, $\Delta \circ [\text{div}] = \dfrac{1}{H_L}$ and $H_I \circ \mathcal{O}\{-\} =$

$\frac{1}{H_L}$. Therefore the groups $\check{H}^1(X, \mathcal{O}^*_{X,alg})$, $Inv(X)$, $LB(X)$, $\text{Pic}(X)$ are isomorphic to each other. Also, the following diagram commutes.

$$
\begin{tikzcd}
 & Inv(X) \arrow[d, "H_I"] & \\
 & \check{H}^1(X, \mathcal{O}^*_{X,alg}) & \\
LB(X) \arrow[uur, "\mathcal{O}\{-\}"] \arrow[ur, "\frac{1}{H_L}"'] \arrow[rr, "{[div]}"] & & Pic(X) \arrow[uul, "\mathcal{O}[-]"'] \arrow[ul, "\Delta"]
\end{tikzcd}
$$

Chapter 4

A Uniqueness Theorem for Algebraic Curves

This chapter is based on the paper [8] and the textbook [3].

4.1 Associated Curves and Normal Forms

Suppose $f : X \to \mathbb{P}^n$ is a non-degenerate holomorphic embedding of a compact Riemann surface X, that is $f(X)$ is not contained in any hyperplane in $\mathbb{P}^n$. Notice that the image, $f(X)$ is an algebraic curve. We can write f locally by $f(z) = [v_0(z) : \ldots : v_n(z)]$ where $v(z) = (v_0(z), \ldots, v_n(z))$ is a holomorphic vector-valued function to $\mathbb{C}^{n+1}$.

Note that $f(z)$ is well-defined even if $v(z) = 0$ at isolated point. Suppose $v(z_0) = 0$ and $k = \min\{\mathrm{ord}_{z_0} v_i(z)\}$. We can write $f(z) = [z^{-k}v_0(z) : \ldots : z^{-k}v_n(z)]$ which is well-defined near z_0.

Definition 4.1 *Define the k-th associated curve of f, $f_k : X \to G(k+1, n+1) \subset P(\bigwedge^{k+1} \mathbb{C}^{n+1})$ by $f_k(z) = [v(z) \wedge v'(z) \wedge \ldots \wedge v^{(k)}(z)]$.*

To check f_k is well-defined, we need to show that f_k is independent of chart z, independent of local representation $v(z)$, and $v(z) \wedge v'(z) \wedge \ldots \wedge v^{(k)}(z) \not\equiv 0$.

Let w be another chart on X. Then $\frac{\partial v}{\partial w} = \frac{\partial v}{\partial z}\frac{\partial z}{\partial w}$. This gives $v \wedge \frac{\partial v}{\partial w} = \frac{\partial z}{\partial w}(v \wedge \frac{\partial v}{\partial z})$. In general, $v \wedge \ldots \wedge \frac{\partial^i v}{\partial w^i} = (\frac{\partial z}{\partial w})^{\frac{i(i+1)}{2}}(v \wedge \ldots \wedge \frac{\partial^i v}{\partial z^i})$. Thus, $[v(w) \wedge \ldots \wedge v^{(k)}(w)] = [v(z) \wedge \ldots \wedge v^{(k)}(z)]$ and f_k is independent of chart. Let $u(z) = p(z)v(z)$ be another local representation of $f(z)$. Then $u(z) \wedge u'(z) = p^2(z)v(z) \wedge v'(z)$. In general, $u(z) \wedge \ldots u^{(i)}(z) = p^{i+1}(z)v(z) \wedge \ldots \wedge v^{(i)}(z)$. Thus, $[u(z) \wedge \ldots \wedge u^{(k)}(z)] = [v(z) \wedge \ldots \wedge v^{(k)}(z)]$ and f_k is independent of local representation. Suppose $v(z) \wedge \ldots \wedge v^{(i)}(z) \equiv 0$ and $v(z) \wedge \ldots \wedge v^{(i-1)}(z) \not\equiv 0$ for some $i \leq k$. So, $v^{(i)}(z)$ can be written as linear combination of $v(z), \ldots, v^{(i-1)}(z)$. Also, $(v(z) \wedge \ldots \wedge v^{(i-1)}(z))' = v(z) \wedge \ldots \wedge v^{(i-2)}(z) \wedge v^{(i)}(z) = p(z)v(z) \wedge \ldots \wedge v^{(i-2)}(z) \wedge v^{(i-1)}(z)$ for some $p(z)$. Hence, $f_{i-1}(z)$ is constant and $f(X)$ lies in a $(i-1)$-plane in $\mathbb{P}^n$, which contradicts to the nondegeneracy of f. Therefore, f_k is well-defined.

Fix $z_0 \in X$. Write $f(z) = [v_0(z) : \ldots : v_n(z)]$ where $v(z_0) \neq 0$. Make a linear change of coordinate in $\mathbb{C}^{n+1}$ such that $v(z_0) = (1, 0, \ldots, 0)$. Write $(v_0(z), \ldots, v_n(z)) = (z - z_0)^{\alpha_1+1}(v_1^1(z), \ldots, v_n^1(z))$ with $(v_1^1(z_0), \ldots, v_n^1(z_0)) \neq 0$ and $\alpha_1 \geq 0$. Next, make a linear change of last n coordinates in $\mathbb{C}^{n+1}$ such that $(v_1^1(z_0), \ldots, v_n^1(z_0)) = (1, 0, \ldots, 0)$. Write $(v_1^1(z), \ldots, v_n^1(z)) = (z - z_0)^{\alpha_2+1}(v_2^2(z), \ldots, v_n^2(z))$ with $(v_2^2(z_0), \ldots, v_n^2(z_0)) \neq 0$ and $\alpha_2 \geq 0$. By continuing the above process, we obtain $v(z) = (1 + \ldots, (z - z_0)^{\alpha_1+1} + \ldots, (z - z_0)^{\alpha_1+\alpha_2+2} + \ldots, \ldots, (z - z_0)^{\alpha_1+\ldots+\alpha_n+n} + \ldots)$. This expression of f is called the normal form of curve f near z_0. We can further assume $z_0 = 0$ and normalize $v(z)$ by making $v_0(z) \equiv 1$. Thus, we obtain $v(z) = (1, z^{\alpha_1+1} + \ldots, z^{\alpha_1+\alpha_2+2} + \ldots, \ldots, z^{\alpha_1+\ldots+\alpha_n+n} + \ldots)$ near $z_0 = 0$.

Note that $f_k(z_0)$ is spanned by first $k+1$ linearly independent vectors from $v(z_0), v'(z_0), v''(z_0), \ldots$. Writing f in the normal form is same as choosing basis in $\mathbb{C}^{n+1}$ such that $f_k(z_0)$ is spanned by $\{e_0, \ldots, e_k\}$.

Definition 4.2 *Given in terms of Euclidean coordinates in a neighbourhood of $f(z_0)$ by $f_1(z), \ldots, f_n(z)$. Define the ramification index of f at z_0 to be $\beta(z_0) =$*

$\min_i\{\mathrm{ord}_{z_0}(\frac{\partial f_i}{\partial z})\}$.

Denote $\beta_k(z_0)$ to be the ramification index of the k-th associated curve of f, f_k at z_0. To compute $\beta_k(z_0)$, we need to write f in normal form. That is $v(z) = (1, z^{\alpha_1+1} + \ldots, z^{\alpha_1+\alpha_2+2} + \ldots, \ldots, z^{\alpha_1+\ldots+\alpha_n+n} + \ldots)$. The homogenous coordinates of $f_k(z)$ in $P(\bigwedge^{k+1}\mathbb{C}^{n+1})$ are the determinants of $(k+1)\times(k+1)$ minors of the $(k+1)\times(n+1)$ matrix

$$\begin{pmatrix} v(z) \\ v'(z) \\ \vdots \\ v^{(k)}(z) \end{pmatrix} = \begin{pmatrix} 1+\ldots & z^{\alpha_1+1}+\ldots & \ldots & z^{\alpha_1+\ldots+\alpha_n+n}+\ldots \\ 0 & (\alpha_1+1)z^{\alpha_1}+\ldots & \ldots & \ldots \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \vdots & \vdots & \vdots \end{pmatrix}$$

Denote $M_{i_1,\ldots,i_{k+1}}$ to be the minor that is composed of $i_1,\ldots,i_{k+1}$ columns. Then $|M_{1,\ldots,k+1}|$ has the smallest order at 0, and $|M_{1,\ldots,k,k+2}|$ has the second smallest order at 0. Therefore, $\beta_k(z_0) = \min_{i_1,\ldots,i_{k+1}}\{\mathrm{ord}_{z_0}(\frac{\partial}{\partial z}\frac{|M_{i_1,\ldots,i_{k+1}}(z)|}{|M_{1,\ldots,k+1}(z)|})\} = \mathrm{ord}_{z_0}(\frac{\partial}{\partial z}\frac{|M_{1,\ldots,k,k+2}(z)|}{|M_{1,\ldots,k+1}(z)|})$.

$$\begin{aligned}
&|M_{1,\ldots,k+1}| \\
&= \begin{vmatrix} 1+\ldots & z^{\alpha_1+1}+\ldots & \ldots & z^{\alpha_1+\ldots+\alpha_k+k}+\ldots \\ 0 & (\alpha_1+1)z^{\alpha_1}+\ldots & \ldots & \ldots \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \vdots & \vdots & \vdots \end{vmatrix} \\
&= \begin{vmatrix} (\alpha_1+1)z^{\alpha_1}+\ldots & (\alpha_1+\alpha_2+2)z^{\alpha_1+\alpha_2+1}+\ldots & \ldots & \ldots \\ \alpha_1(\alpha_1+1)z^{\alpha_1-1}+\ldots & (\alpha_1+\alpha_2+1)(\alpha_1+\alpha_2+2)z^{\alpha_1+\alpha_2}+\ldots & \ldots & \ldots \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \end{vmatrix}
\end{aligned}$$

$$=z^{\alpha_1}z^{\alpha_1-1}\cdots z^{\alpha_1-(k-1)}\times$$

$$\begin{vmatrix} (\alpha_1+1)+\ldots & (\alpha_1+\alpha_2+2)z^{\alpha_2+1}+\ldots & \ldots & \ldots \\ \alpha_1(\alpha_1+1)+\ldots & (\alpha_1+\alpha_2+1)(\alpha_1+\alpha_2+2)z^{\alpha_2+1}+\ldots & \ldots & \ldots \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \end{vmatrix}$$

$$=z^{k\alpha_1-\frac{(k)(k-1)}{2}}z^{\alpha_2+1}z^{\alpha_3+2}\ldots z^{\alpha_k+(k-1)}\times$$

$$\begin{vmatrix} (\alpha_1+1)+\ldots & (\alpha_1+\alpha_2+2)+\ldots & \ldots & \ldots \\ \alpha_1(\alpha_1+1)+\ldots & (\alpha_1+\alpha_2+1)(\alpha_1+\alpha_2+2)+\ldots & \ldots & \ldots \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \end{vmatrix}$$

$$=z^{k\alpha_1+\alpha_2+\ldots+\alpha_k}\begin{vmatrix} (\alpha_1+1)+\ldots & (\alpha_1+\alpha_2+2)+\ldots & \ldots & \ldots \\ \alpha_1(\alpha_1+1)+\ldots & (\alpha_1+\alpha_2+1)(\alpha_1+\alpha_2+2)+\ldots & \ldots & \ldots \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \end{vmatrix}$$

Similarly, we have $|M_{1,\ldots,k,k+2}|$

$$= z^{k\alpha_1+\alpha_2+\ldots+\alpha_k+\alpha_{k+1}+1}\times$$

$$\begin{vmatrix} (\alpha_1+1)+\ldots & (\alpha_1+\alpha_2+2)+\ldots & \ldots & \ldots \\ \alpha_1(\alpha_1+1)+\ldots & (\alpha_1+\alpha_2+1)(\alpha_1+\alpha_2+2)+\ldots & \ldots & \ldots \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \end{vmatrix}$$

Therefore, we have $\beta_k(z_0) = \mathrm{ord}_{z_0}(\frac{\partial}{\partial z}\frac{|M_{1,\ldots,k,k+2}(z)|}{|M_{1,\ldots,k+1}(z)|}) = \alpha_{k+1}$. We denote $\beta_k = \sum_{z\in X}\beta_k(z)$.

4.2 Proof of a Uniqueness Theorem for Algebraic Curves

Theorem 4.3 (The General Plücker Formula) *Let X be a compact Riemann surface of genus g, and $f : X \to \mathbb{P}^n$ be a non-degenerate holomorphic embedding. Denote d_k to be the number of intersections of $f_k(X)$ meeting a $(n-k-1)$-plane in $\mathbb{P}^n$. Then $2g-2 = -2d_k + \beta_k + d_{k-1} + d_{k+1}$ for $1 \leq k \leq n-1$.*

Proof Please refer to [3]. □

Proposition 4.4 *Let X be a compact Riemann surface of genus g, and $f : X \to \mathbb{P}^n$ be a non-degenerate holomorphic embedding. Let $H_1, H_2, \ldots, H_q$ be hyperplanes in $\mathbb{P}^n$ in general position, that is any $n+1$ of them are linearly independent. Let $M = \bigcup_{i=1}^{q} f^{-1}(H_i)$. Then $(q-(n+1))d_0 \leq \frac{1}{2}n(n+1)\{2(g-1)+|M|\}$ where $|M|$ denote the cardinality of M.*

Proof Since f is non-degenerate, $f(X)$ is not contained in any hyperplane. We can write $M = \{p_1, \ldots, p_r\}$. If H_i intersects $f(X)$ at $p_j \in M$ with multiplicity m_{ij}, then by the definition of d_0, for each hyperplane, $\sum_{1\leq j\leq r} m_{ij} = d_0$. Thus,

$$\sum_{1\leq i\leq q}\sum_{1\leq j\leq r} m_{ij} = d_0 q \tag{1}$$

For all $p_j \in M$, since $H_1, H_2, \ldots, H_q$ are in general position, at most n hyperplanes can intersect $f(X)$ at p_j. Then there exists $A \subset \{1, 2, \ldots, q\}$ with $|A| = n$ such that

$$\sum_{1\leq i\leq q} m_{ij} \leq \sum_{i\in A} m_{ij} \tag{2}$$

Suppose $z(p_j) = 0$ and write f in normal form at $z = 0$, that is $f(z) = [1+\cdots : z^{\alpha_1+1}+\cdots : \ldots : z^{\alpha_1+\cdots+\alpha_n+n}+\cdots]$. For all p_j, for a unique hyperplane, it can intersect $f(X)$ at p_j with multiplicity at most $\alpha_1+\cdots+\alpha_n+n$. The second hyperplane can intersect $f(X)$ at p_j with multiplicity at most $\alpha_1+\cdots+\alpha_{n-1}+n-1$, and so on. So, we have

$$\sum_{i\in A} m_{ij} \leq (\alpha_1+1)+(\alpha_1+\alpha_2+2)+\cdots+(\alpha_1+\cdots+\alpha_n+n)$$

$$= \sum_{1\leq i\leq n}(n+1-i)\alpha_i(p_j)+\frac{1}{2}n(n+1) \tag{3}$$

Combine (1), (2), (3), we have

$$\begin{aligned}
d_0 q &= \sum_{1\leq j\leq r}\sum_{1\leq i\leq q} m_{ij} \qquad by(1)\\
&\leq \sum_{1\leq j\leq r}\sum_{i\in A} m_{ij} \qquad by(2)\\
&\leq \sum_{1\leq j\leq r}[\sum_{1\leq i\leq n}(n+1-i)\alpha_i(p_j)+\frac{1}{2}n(n+1)] \qquad by(3)\\
&\leq \sum_{1\leq i\leq n}(n+1-i)\sum_{1\leq j\leq r}\alpha_i(p_j)+\frac{r}{2}n(n+1)\\
&\leq \sum_{1\leq i\leq n}(n+1-i)\beta_{i-1}+\frac{r}{2}n(n+1)
\end{aligned}$$

By the general Plücker formula, we have $2g-2=-2d_k+\beta_k+d_{k-1}+d_{k+1}$ for $1\leq k\leq n-1$ and $2g-2=-2d_0+\beta_0+d_1$. Hence $\sum_{1\leq i\leq n}(n+1-i)\beta_{i-1}$

$$\begin{aligned}
&= \sum_{2\leq i\leq n}[(n+1-i)(2d_{i-1}-d_i-d_{i-2}+2g-2))]+n\beta_0\\
&= [-(n-1)d_0+nd_1-d_n]+n(n-1)(g-1)+[2nd_0-nd_1+2n(g-1)]\\
&= n(n+1)(g-1)+(n+1)d_0
\end{aligned}$$

Therefore, we have

$$d_0 q \leq n(n+1)(g-1)+(n+1)d_0+\frac{1}{2}n(n+1)|M|$$

implies

$$(q-(n+1))d_0 \leq \frac{1}{2}n(n+1)\{2(g-1)+|M|\}$$

□

Lemma 4.5 *Let X be a compact Riemann surface of genus g, and $f, g : X \to \mathbb{P}^n$ be two distinct non-degenerate holomorphic embeddings. Let $H_1, H_2, \ldots, H_q$ be hyperplanes in general position in $\mathbb{P}^n$. Then there exists a dense set $S \subset \mathbb{C}^{n+1} - \{0\}$ such that for all $s = (s_0, \ldots, s_n) \in S$, hyperplane H_s defined by $s_0 z_0 + \ldots + s_n z_n = 0$ satisfies $f^{-1}(H_i) \bigcap f^{-1}(H_s) = \emptyset$ and $g^{-1}(H_i) \bigcap g^{-1}(H_s) = \emptyset$ for all $1 \le i \le q$.*

Proof Let $A = \bigcup_{i=1}^{q} (f^{-1}(H_i) \cup g^{-1}(H_i))$ which is a finite subset of X. Suppose $(v_0, \ldots, v_n)$ and $(u_0, \ldots, u_n)$ are local representations of f and g respectively. For all $p \in A$, let $K_{f,p} = \{(z_0, \ldots, z_n) \in \mathbb{C}^{n+1} : z_0 v_0(p) + \ldots + z_n v_n(p) = 0\}$ and $K_{g,p} = \{(z_0, \ldots, z_n) \in \mathbb{C}^{n+1} : z_0 u_0(p) + \ldots + z_n u_n(p) = 0\}$. Define $K = \bigcup_{p \in A} (K_{f,p} \cup K_{g,p})$. Then the dense set $S = \mathbb{C}^{n+1} - K$ is our desired set. □

Theorem 4.6 (Uniqueness Theorem for Algebraic Curves) *Let X be a compact Riemann surface of genus g, and $f, g : X \to \mathbb{P}^n$ be two distinct non-degenerate holomorphic embeddings. Let $H_1, H_2, \ldots, H_q$ be hyperplanes in general position in $\mathbb{P}^n$. Suppose that*

1. *$f^{-1}(H_i) = g^{-1}(H_i)$ for all $1 \le i \le q$*
2. *$f = g$ on $\bigcup_{i=1}^{q} f^{-1}(H_i)$.*

Then $q \le \frac{1}{2}\{(n+1)^2 + \sqrt{(n+1)^4 + 4n^2(n+1)(g-1)}\}$.

Proof Let $d_1 = \deg(f)$ and $d_2 = \deg(g)$. WLOG, assume $2 \le d_1 \le d_2$. Let $H_1, \ldots, H_q$ given by $H_i : a_{i_0} z_0 + \cdots + a_{i_n} z_n = 0$. Let $(v_0, \ldots, v_n)$ and $(u_0, \ldots, u_n)$ be the local representations of f and g respectively. Let $(f, H_i) = a_{i_0} v_0 + \cdots + a_{i_n} v_n$ and $(g, H_i) = a_{i_0} u_0 + \cdots + a_{i_n} u_n$. By Lemma 4.5, there exists a hyperplane H_s such that $f^{-1}(H_i) \bigcap f^{-1}(H_s) = \emptyset$ and $g^{-1}(H_i) \bigcap g^{-1}(H_s) = \emptyset$ for $1 \le i \le q$. For each i, $\dfrac{(f, H_i)}{(f, H_s)} - \dfrac{(g, H_i)}{(g, H_s)}$ is a well-defined rational function on X, because it is independent of the choice of local representation.

We want to show that there exits $1 \leq i_0 \leq q$ such that $\frac{(f,H_{i_0})}{(f,H_s)} - \frac{(g,H_{i_0})}{(g,H_s)} \not\equiv 0$. Suppose not, that is $\frac{(f,H_i)}{(f,H_s)} - \frac{(g,H_i)}{(g,H_s)} \equiv 0$ for all $1 \leq i \leq q$. By changing homogenous coordinates in $\mathbb{P}^n$, we may assume that $H_s : z_0 = 0$ and $H_i : z_i = 0$ for all $1 \leq i \leq n$. Then $\frac{v_i}{v_0} = \frac{u_i}{u_0}$ for all $1 \leq i \leq n$. Thus $f \equiv g$, which gives a contradiction.

Let $M = \bigcup_{i=1}^q f^{-1}(H_i)$. $f = g$ on M by assumption (2). For all $x \in M$, x is a zero of $\frac{(f,H_{i_0})}{(f,H_s)} - \frac{(g,H_{i_0})}{(g,H_s)}$. Hence

$$|M| \leq \deg\left(\frac{(f,H_{i_0})}{(f,H_s)} - \frac{(g,H_{i_0})}{(g,H_s)}\right) \leq 2d_2 \tag{4}$$

By assumption (1), $M = \bigcup_{i=1}^q f^{-1}(H_i) = \bigcup_{i=1}^q g^{-1}(H_i)$. By Proposition 4.4, we have

$$(q-(n+1))d_2 \leq \frac{1}{2}n(n+1)\{2(g-1)+|M|\} \tag{5}$$

Combine (4) and (5),

$$q \leq \frac{1}{d_2}n(n+1)(g-1)+(n+1)^2 \tag{6}$$

Also, $x \in M$ is a zero of $\frac{(f,H_{i_0})}{(f,H_s)} - \frac{(g,H_{i_0})}{(g,H_s)}$ with $\mathrm{mult}_x\left(\frac{(f,H_{i_0})}{(f,H_s)} - \frac{(g,H_{i_0})}{(g,H_s)}\right) \geq \min\{\mathrm{mult}_x\frac{(f,H_{i_0})}{(f,H_s)}, \mathrm{mult}_x\frac{(g,H_{i_0})}{(g,H_s)}\}$. If $f^{-1}(H_i)$ has one element, then $\mathrm{mult}_x \left(\frac{(f,H_{i_0})}{(f,H_s)} - \frac{(g,H_{i_0})}{(g,H_s)}\right) \geq d_1 \geq 2$. If $f^{-1}(H_i)$ has more than one element, it has at least two zeros of the function $\frac{(f,H_{i_0})}{(f,H_s)} - \frac{(g,H_{i_0})}{(g,H_s)}$. By summing up all the hyperplanes,

$$\begin{aligned}
2q &\leq \sum_{i=1}^{q}\sum_{x\in f^{-1}(H_i)} \mathrm{mult}_x\left(\frac{(f,H_{i_0})}{(f,H_s)} - \frac{(g,H_{i_0})}{(g,H_s)}\right) \\
&\leq n\sum_{x\in f^{-1}(H_i)} \mathrm{mult}_x\left(\frac{(f,H_{i_0})}{(f,H_s)} - \frac{(g,H_{i_0})}{(g,H_s)}\right) \\
&\leq n\deg\left(\frac{(f,H_{i_0})}{(f,H_s)} - \frac{(g,H_{i_0})}{(g,H_s)}\right) \\
&\leq 2nd_2
\end{aligned}$$

Hence,

$$q \leq nd_2 \tag{7}$$

Combine (6) and (7), $q \leq \min\{nd_2, (n+1)^2 + \frac{1}{d_2}n(n+1)(g-1)\}$. With respect to d_2, $\min\{nd_2, (n+1)^2 + \frac{1}{d_2}n(n+1)(g-1)\}$ is maximum when $nd_2 = (n+1)^2 + \frac{1}{d_2}n(n+1)(g-1)$. This gives $d_2 = \frac{1}{2n}[(n+1)^2 + \sqrt{(n+1)^4 + 4n^2(n+1)(g-1)}]$ Therefore, $q \leq \frac{1}{2}\{(n+1)^2 + \sqrt{(n+1)^4 + 4n^2(n+1)(g-1)}\}$. □

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